Metric Spaces / Baire Category

Thm: A nonempty product space $\prod_{\alpha \in I} M_\alpha$ is metrizable iff each M_α is metrizable and M_α is a single point for all but countably many of the indices.

Pf: If $\prod M_\alpha$ is metrizable, then each factor space M_α is homeomorphic to a subspace of the product and hence, since subspaces of metric spaces are themselves metric spaces, they are metrizable.

Conversely, let $\{M_n\}_{n \in \omega}$ be metric spaces. Each is metrizable via a bounded metric, ρ_n st $|\rho_n(x,y)| \leq 1$ $\forall x, y \in M_n$.

for $\vec{x}, \vec{y} \in \prod_{n=1}^{\infty} M_n$, define $\rho(\vec{x},\vec{y}) = \sum_{n=1}^{\infty} \frac{\rho_n(x_n, y_n)}{2^n}$ we claim $\rho(\vec{x},\vec{y})$ is a metric. Clearly positive semidefinite, and clearly symmetric by symmetry of each ρ_n. For triangle inequality, $\rho_n(x_n,y_n) \leq \rho_n(x_n,z_n) + \rho_n(z_n,y_n) \Rightarrow \sum_{n=1}^{\infty} \frac{\rho_n(x_n,y_n)}{2^n} \leq \sum_{n=1}^{\infty} \frac{\rho_n(x_n,z_n)}{2^n} + \sum_{n=1}^{\infty} \frac{\rho_n(z_n,y_n)}{2^n}$

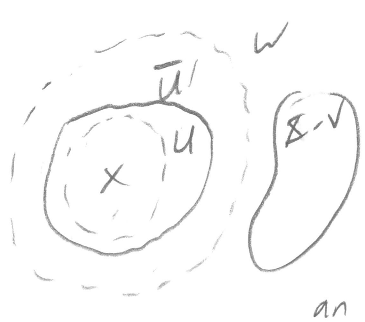
$\Rightarrow \rho(\vec{x},\vec{y}) \leq \rho(\vec{x},\vec{z}) + \rho(\vec{z},\vec{y})$. NTS that it induces the product topology. Let $\vec{x} \in \prod_{n=1}^{\infty} M_n$, U a basic nbhd of $\vec{x}$, ie

$U = \bigcap_{j=1}^{m} \pi_j^{-1}(B_{\rho_j}(x_j, \varepsilon_j))$. Let $\varepsilon = \min\{\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2^2}, \ldots, \frac{\varepsilon_m}{2^m}\}$.

Ugh I'm not finishing this. $\boxed{\ }$

Urysohn's Metrization Thm: A regular, 2^{nd} countable space is metrizable. (ie T_3 + 2^{nd} ctble $\Rightarrow$ Metrizable / T_4 / T_6)

Pf: Let $\mathcal{B} = \{B_n\}_{n\in\omega}$ be a base for X. Let $\mathcal{A} = \{(U,V) : U, V \in \mathcal{B}$ and $\overline{U} \subseteq V\}$. Since X is T_3, $\forall V \in \mathcal{B}$, $\exists S \in \tau$ st $\overline{S} \subseteq V$ and thus a $U \in \mathcal{B}$ st $U \subseteq S \Rightarrow \overline{U} \subseteq \overline{S}$ $\Rightarrow \overline{U} \subseteq V$, so $\mathcal{A}$ is at least as big as $\mathcal{B}$, and further no bigger than $\mathcal{B} \times \mathcal{B}$. ie $\mathcal{A}$ is countable. Note that 2^{nd} ctble $\Rightarrow$ Lindelöf, and any regular Lindelöf space is normal. So X is normal. Note that since $\overline{U} \subseteq V$, $\overline{U} \cap (X - V) = \emptyset$, and both are closed, so by normality and Urysohn's Lemma, $\exists$ for each pair $(U,V) \in \mathcal{A}$, a cts function $f_{UV} : X \to I$ st $f(\overline{U}) = 0$ and $f(X - V) = 1$. Consider the collection $\mathcal{F} = \{f_{UV}\}$. This is a countable collection. Further, if $x \in X$, $X - V$ closed, then V is an open nbhd of x, ie $\exists W \in \mathcal{B}$ st $x \in W \subseteq V$, and further a $U \in \mathcal{B}$ st $x \in U \subseteq \overline{U} \subseteq W \subseteq X - V$. Thus the function $f_{UV}(\overline{U}) = 0$, ie $f_{UV}(x) = 0$. and $f_{UV}(X - V) = 1$,



So $\mathcal{F}$ separates points from closed sets. Thus the evaluation map $e : X \to \prod_{f \in \mathcal{F}} I_f$ is an embedding, ie X is homeomorphic to a subspace of $\prod_{f \in \mathcal{F}} I_f$. But since $\mathcal{F}$ is countable, and each I_f is a metric space, $\prod I_f$ is itself a metric space

Thus X is homeomorphic to a metric space and is therefore metrizable. $\boxed{\#}$

Def^n: A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space is *Cauchy* if $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ st $\rho(x_n, x_m) < \varepsilon\ \forall m,n \geq N$.

A metric space X is *complete* if every Cauchy sequence converges.

We say ρ is a *complete metric* for X, and call X *completely metrizable* if there $\exists$ a complete metric which generates it.

☆ Complete metrizability *is* a topological property, but completeness itself is a property of a metric.

Ex: $((0,1), \rho_{std})$ is not a complete metric space*, but $(0,1)$ *is* completely metrizable, since it is homeomorphic to $\mathbb{R}$.

(*) $\{x_n\} = \frac{1}{n}$ is a Cauchy sequence, but does not converge in the usual metric.

Ie, a subspace of a metric space is metrizable by the restriction of the old metric, but that doesn't mean that there isn't an equivalent metric which *is* complete.

The *completion* of a metric space (X, ρ) is a complete metric space (Y, σ) st X can be embedded into Y, and st the image of X is dense in Y.

Metric spaces (M, ρ) and (N, σ) are *isometric* if $\exists$ a distance preserving injective function $f: M \to N$. That is $\forall x,y \in M$, $\rho(x,y) = \sigma(f(x), f(y))$.

Thm: Every metric space M can be isometrically embedded as a dense subset of a complete space. The resulting completion is unique up to isometry.

Pf: Let (M, ρ) be a metric space, $\mathcal{M}$ be the set of all cauchy sequences in M. Note that if $\{x_n\}, \{y_n\} \in \mathcal{M}$, then $\rho(x_n, y_n)$ is also a cauchy sequence in $\mathbb{R}$, which is complete, thus, defining

$$d(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} \rho(x_n, y_n)$$

is legal. We claim d is a pseudometric. Clearly, $d(\{x_n\}, \{y_n\}) = 0$ if $\{x_n\} = \{y_n\}$. Clearly symmetric by symmetry of ρ, and clearly satisfies the triangle inequality since so does ρ and limits distribute over sums. Let $(\mathcal{M}^*, d^*)$ is the metric space induced by the pseudometric d, where $\mathcal{M}^*$ is the collection of equivalence classes $[\{x_n\}]$ where $\{x_n\} \sim \{y_n\}$ if $d(\{x_n\}, \{y_n\}) = 0$ (This forces $d^*([\{x_n\}], [\{y_n\}])$ $:= d(\{x_n\}, \{y_n\})$ where $\{x_n\} \in [\{x_n\}]$, $\{y_n\} \in [\{y_n\}]$ to be a true metric) Let $g: M \to \mathcal{M}^*$ be defined by $g(x) = [\{x, x, x, x, \ldots\}]$

we claim g is an isometry. Let $x, y \in M$. Then

$$d^*(g(x), g(y)) = \lim_{n\to\infty} \rho(x, y) = \rho(x, y),$$ confirming this claim.

I don't feel like I'm getting much out of this proof, so I'm going to skip showing that $g(M)$ is dense in $\mathcal{M}^*$ or that $\mathcal{M}^*$ is complete or that $\mathcal{M}^*$ is unique up to isometry. $\boxed{\equiv}$

$\boxed{\text{Corollary: Any metric space has a completion}}$

Recall, a set A is <u>nowhere dense</u> if $\bar{A}^{\circ} = \varnothing$. Also recall that the collection of dense sets form an ideal

<u>Def^n</u>: A space X is a <u>Baire Space</u> if, given a countable collection of closed nowhere dense sets $\{A_n\}_{n\in\omega}$, $\left(\bigcup A_n\right)^{\circ} = \varnothing$.

A subspace A of a space X is said to be of the <u>1st category</u> in X if it is contained in the union of a ctble collection of closed, nwd sets. Otherwise, A is of the <u>2nd category</u>.

Thus, X is a Baire space iff $\forall$ nonempty open sets U in X, U is of the second category, since the union of a ctble collection of closed nwd sets has empty interior iff it contains no nonempty open sets iff no nonempty open sets are contained in the union of a ctble collection of closed nwd sets.

<u>Lemma</u>: X is a Baire space iff $\forall$ ctble collections of dense open sets $\{U_n\}_{n\in\omega}$ in X, $\bigcap U_n$ is also dense in X.

[Ie X is a Baire space iff ctble unions of small sets are small iff ctble intersections of large sets are large]

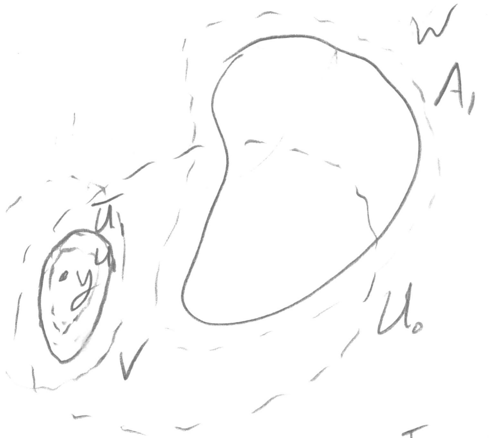
<u>Pf</u>: Let $\{U_n\}_{n\in\omega}$ be open, dense then $\bigcap U_n$ is dense in X iff $\bigcup U_n^c$ is nwd, $\boxed{\text{Pf}}$ (La 2.8)

<u>Thm (Baire Category Thm)</u>: If X is a compact Hausdorff space or a complete metric space, then X is Baire.

<u>Pf</u>: Let $\{A_n\}_{n\in\omega}$ be a ctble collection of closed sets with empty interiors. NTS $\forall U\in\tau$, $U\neq\emptyset$, $\exists x\in U$ st $x\notin\bigcup A_n$.

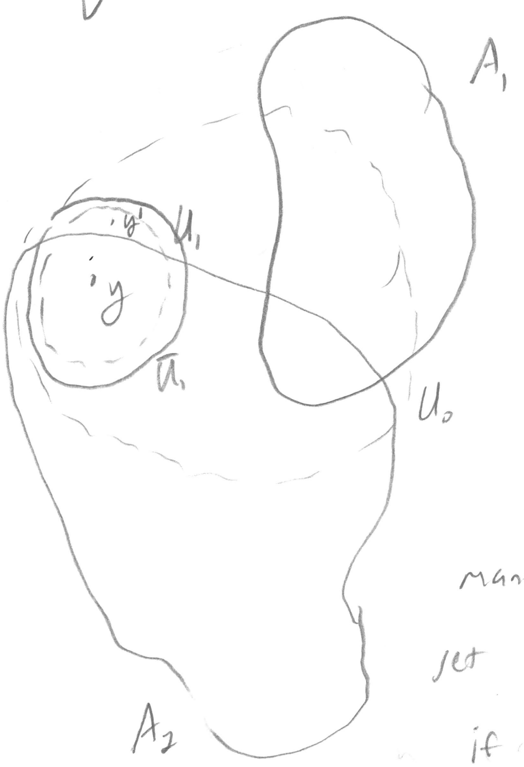
$\mathcal{S}$ first that X is compact, Hausdorff. Then X is T_4 and more importantly regular, so since $A_1^\circ=\emptyset$, we may fix any nonempty open $U_0\in\tau$, and choose a $y\in U_0$ st $y\notin A_1$.

Then since A_1 is closed, $y\notin A_1$, regularity allows us to let $U_1\in\tau$ be st $y\in U_1$, $\overline{U_1}\cap A_1=\emptyset$ [$\exists V\ W$ st $V\cap W=\emptyset$, $y\in V$ $A_1\subseteq W$, then



Then $y \in V \cap U_0 \neq \emptyset$ and $V \cap U_0 \in \mathcal{T} \Rightarrow$ $\exists$ a shrinking U_1 st $y \in U_1 \subseteq \overline{U}_1 \subseteq V \cap U_0 \subseteq U_0$, and $V \cap W = \emptyset \Rightarrow V \cap A_1 = \emptyset \Rightarrow (V \cap U_0) \cap A_1 = \emptyset$ $\therefore$ $\overline{U}_1 \subseteq V \cap U_0 \Rightarrow \overline{U}_1 \cap A_1 = \emptyset$]

To summarize, we have A_1 closed, $y \notin A_1$, $y \in U_0$, $y \in \overline{U}_1 \subseteq U_0$ with $\overline{U}_1 \cap A_1 = \emptyset$. If X is a metric space, let WLOG $\operatorname{diam}(U_1) = \sup\{\rho(x_1, x_2) : x_1, x_2 \in U_1\} < 1$.



Next, since $U_1 \not\subseteq A_2$, $\exists y'$ st $y' \notin A_2$ and $y' \in U_1$. By the same argument, $\exists U_2 \in \mathcal{T}$ st $y' \in U_2$, $\overline{U}_2 \subseteq U_1$, $\overline{U}_2 \cap A_2 = \emptyset$, and if X a metric space, $\operatorname{diam}(U_2) < \frac{1}{2}$. Continue in this manner to define for each $n \in \omega$ a nonempty open set U_n st $\overline{U}_n \subset U_{n-1}$, $\overline{U}_n \cap A_n = \emptyset$, and if X a metric space, $\operatorname{diam}(U_n) < \frac{1}{n}$.

We claim $\bigcap \overline{U}_n \neq \emptyset$. If we can show this, we're done, since then if $x \in \bigcap \overline{U}_n$, then by construction $\forall n \in \omega$, $x \in U_n \wedge \overline{U}_n \cap A_n = \emptyset$, and moreover $\overline{U}_n \subseteq U_0 \Rightarrow x \in U_0$, which was arbitrary, and so $x \notin \bigcup A_n$. If X is compact, T_2, then the sequence of closed sets $\{\overline{U}_n\}$ is nested and has the finite intersection property, since if $m = \max\{n_i\}_{i=1}^{k}$ for a subcollection $\{\overline{U}_{n_1}, \ldots, \overline{U}_{n_k}\}$, then U_m is nonempty $\Rightarrow x \in U_m$ $\Rightarrow x \in \bigcap \overline{U}_m$

So since X is compact the intersection $\bigcap \overline{U_n}$ is nonempty.

If X is a complete metric space, for each n, choose $x_n \in \overline{U_n}$ st $x_n \notin \overline{U_m}$ $\forall m > n$. [can do since $diam(U_m) < diam(U_n)$ $\Rightarrow$ $\exists x, y \in U_n$ st $\rho(x,y) > \sup\{\rho(a,b) : a, b \in U_m\} \Rightarrow x \notin diam(U_m)$ or $y \notin diam(U_n)$]. Then the sequence $\{x_n\}$ is cauchy. Since $\rho(x_n, x_m) \leq \max\{\frac{1}{n}, \frac{1}{m}\}$. For $\varepsilon > 0$, choose N st $\frac{1}{N} < \varepsilon$, then if $m, n > N$, $\rho(x_n, x_m) < \frac{1}{N} < \varepsilon$. Since X is complete, the sequence $\{x_n\}$ converges to some x, ie $x \in \overline{U_n}$ $\forall n$, ie $x \in \bigcap \overline{U_n}$, ie $\bigcap \overline{U_n}$ is nonempty. $\boxed{\Xi}$

Ex $\boxed{E \text{ is a Baire Space.}}$

Pf: Note that $\mathbb{R}$ is a complete metric space and thus Baire. Note that if D is dense in E, then it's dense in $\mathbb{R}_{std}$. Since E is finer than $\mathbb{R}_{std}$, so if all open sets in E intersect D then certainly all open sets in $\mathbb{R}_{std}$ intersect D. Further, if D is dense in $\mathbb{R}_{std}$, then for any open set U in E, $\exists a, b \in \mathbb{R}$ st $[a,b) \subseteq U$, $a < b$, so if we let then $(a,c) \subseteq [a,b)$, and (a,c) open in $\mathbb{R}_{std} \Rightarrow (a,c) \cap D \neq \emptyset$ $\Rightarrow [a,b) \cap D \neq \emptyset$. So D is dense in $\mathbb{R}_{std}$ iff D is dense in E.

Then if $\{U_n\}_{n\in\omega}$ is a ctble collection of open dense sets in E, then it is a collection of dense sets in $\mathbb{R}_{std}$. We claim that $int_{\mathbb{R}_{std}}(U_n) := V_n$ is dense in $\mathbb{R}_{std}$. Let $(a,b) \subseteq \mathbb{R}$, then $\exists c, d$ st $[c,d) \subseteq (a,b) \Rightarrow [c,d) \cap U_n \neq \emptyset$, open in E, So $\exists e, f$ st $[e,f) \subseteq [c,d) \cap U_n \Rightarrow (e,f) \subseteq [c,d) \cap U_n \subseteq U_n$, ie $(e,f) \subseteq \text{int}_{\mathbb{R}_{std}}(U_n)$, $(e,f) \subseteq (a,b) \Rightarrow (a,b) \cap \text{int}_{\mathbb{R}_{std}}(U_n) \neq \emptyset$ so $\text{int}_{\mathbb{R}_{std}}(U_n) := V_n$ is dense in $\mathbb{R}_{std}$. Thus, $\bigcap V_n$ is dense in $\mathbb{R}_{std}$ since $\mathbb{R}_{std}$ is a Baire space. But then $\bigcap V_n$ is dense in E, as we've shown. So E is a Baire Space. $\boxtimes$