Connectedness

<u>Def</u>: A space X is <u>disconnected</u> if $\exists$ disjoint nonempty open sets A, B st $A \cup B = X$. We say X is disconnected by A and B. If there does not exist a disconnection, then we say X is <u>connected</u>.

Note: A and B are clopen, since $A = X - B$, $B = X - A$. Further, if X has a nontrivial clopen set E, then $X - E$ is also clopen, and $X = E \cup (X - E)$, so thus equivalently X is connected iff it has a nontrivial clopen set.

<u>Ex</u>: E is disconnected, since basic open sets are clopen.

<u>Ex</u>: For any X w/ $|X| > 1$, $(X, \mathcal{P}(X))$ is disconnected, since all subsets are clopen.

<u>Ex</u>: Let (X, τ) be T_1, and suppose $\exists x \in X$ st $\{x\} \in \tau$. Then $\{x\}$ is clopen, so X is not connected. In particular, $[0, \omega_1)$ and $[0, \omega_1]$ are disconnected, since $\forall \alpha < \omega_1$, $(\alpha - 1, \alpha + 1)$ is open $= \{\alpha\}$.

<u>Ex</u>: $[0,1]_{std}$ is connected. Pf: Suppose I is disconnected by A, B, wLOG $1 \in A$. Then A contains a nbhd of 1, ie A open $\Rightarrow \exists (a,b)$ st $1 \in (a,b) \cap [0,1] = (a,1] \subseteq A$. Since $A \cap B = \emptyset$, we have that $d \le a < 1$ $\forall d \in B$, so $c := \sup(B) \neq 1$. Let $(\alpha, \beta) \subseteq [0,1]$ be a nbhd of c. Then $c < \beta \Rightarrow \beta \notin B \Rightarrow \beta \in A \Rightarrow A \cap (\alpha, \beta) \neq \emptyset$.

Since B is open, if $c \in B$ then $\exists (\alpha, \beta)$ st $c \in (\alpha, \beta) \subseteq B$.

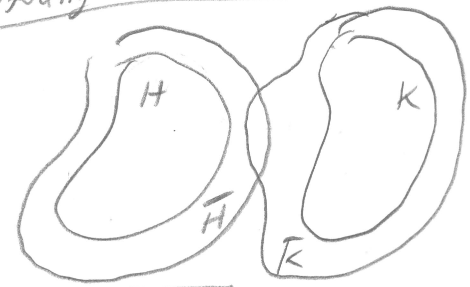
But we showed that $(\alpha,\beta)\cap A \neq \emptyset$, so $A\cap B \neq \emptyset$ $\#$.

Thus, we must conclude $c\in A$. But then if $c\in A$, $\exists(\alpha,\beta)$ st $c\in(\alpha,\beta)\subseteq A$. But then c cant be the sup of B, since $\alpha<c$, and $x\leq c$ $\forall x\in B$ combined w/ $A\cap B=\emptyset$ $\Rightarrow$ everything in (α, c) is smaller than c but larger then everything in B, so $c\neq \sup(B)$, $\#$. Thus $c\notin A$ and $c\notin B$, ie $A\cup B\neq[0,1]$ so there cannot be a disconnection. $\blacksquare$

$\boxed{\text{Thm: The continuous image of a connected space is connected.}}$

Pf: let X connected, $f: X\to Y$ cts, onto. $\not\exists$ Y were disconnected, ie $\exists A, B$ clopen st $A\cup B = Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are clopen in X, and $f^{-1}(A)\cup f^{-1}(B)$ $= f^{-1}(A\cup B) = f^{-1}(Y) = X$, $\#$. $\blacksquare$

Def^{n}: Sets H and K in X are mutually disconnected if $H\cap\bar{K}=\emptyset$ and $\bar{H}\cap K=\emptyset$.



Thm: A subspace E of X is connected iff there are no nonempty mutually seperated sets H and K st $E = H\cup K$

Pf: If E is disconnected by H and K, then $H\cap Cl_X(K)$ $=(H\cap E)\cap Cl_X(K)$ (since $H\cup K=E \Rightarrow H\subseteq E \Rightarrow H\cap E = H$)

$= H\cap(E\cap Cl_X(K)) = H\cap Cl_E(K) = H\cap K$ since K clopen in E.

$=\emptyset$

Similarly, $Cl_X(H) \cap K = \emptyset$. So H and K are mutually seperated.

Conversely, if H and K are mutualy seperated in X and $E = H \cup K$, then $Cl_E(H) = E \cap Cl_X(H) = (H \cup K) \cap Cl_X(H)$

$$= H \cap Cl_X(H) \cup K \cap Cl_X(H)$$

$$= H \cup \emptyset = H. \text{ So } H \text{ is closed.}$$

Similarly, K is rel. closed, so since $H = E - K$ and $K = E - H$, H and K are rel. open, so rel. clopen, so E is disconnected. $\boxed{\Xi}$

Corollary: If H and K are mutualy separated in X and E is a connected subset of $H \cup K$, then $E \subseteq H$ or $E \subseteq K$

Pf: If H and K are mutually separated in X, then

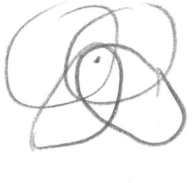
$$\overline{(H \cap E)} \cap (K \cap E) \subseteq (H \cap E) \cap \overline{K} \cap \overline{E} = (H \cap \overline{K}) \cap \overline{E}$$

$$= \emptyset \cap \overline{E} = \emptyset.$$

Similarly, $\overline{H \cap E} \cap (K \cap E) = \emptyset$, so $H \cap E$ and $K \cap E$ are mutually seperated. Since E is connected, it must be that one of these is empty. ie either $H \cap E = \emptyset \Rightarrow K \cap E = E$ $\Rightarrow E \subseteq K$, or $K \cap E = \emptyset \Rightarrow E \subseteq H$. $\boxed{\Xi}$

Thm: If $X = \bigcup X_\alpha$ where each X_α is connected and $\bigcap X_\alpha \neq \emptyset$, then X is connected.

$\swarrow \bigcap X_\alpha \neq \emptyset$



Pf: $\mathcal{S}$ $X = H \cup K$ where H, K are mutually seperated in X. Then since X_α is a connected subset of $H \cup K$, we must have $X_\alpha \subseteq H$ or $X_\alpha \subseteq K$ $\forall \alpha \in I$.

But $\forall \beta \neq \alpha$, $X_\beta \cap X_\alpha \neq \emptyset$ so if $X_\alpha \subseteq H$ and $X_\beta \subseteq K$, then $X_\alpha \cap X_\beta = \emptyset$, $\#$, so either H or K contain all of the X_α. But then $X = \bigcup X_\alpha \subseteq H$ (WLOG) so $K = \emptyset$. Thus X can never be the union of two nonempty mutually disconnected sets i.e. X is connected. $\blacksquare$

$\boxed{\text{Corollary: } \mathbb{R} \text{ is connected.}}$

Pf: $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$, and each set $[-n,n]$ is homeomorphic to $[0,1]$, which we showed was connected. Since $[-1,1] \subseteq [-n,n]$ $\forall n \in \mathbb{N}$, $\bigcap [-n,n] \neq \emptyset$, so by the prvs thm, $\mathbb{R}$ is connected. $\blacksquare$

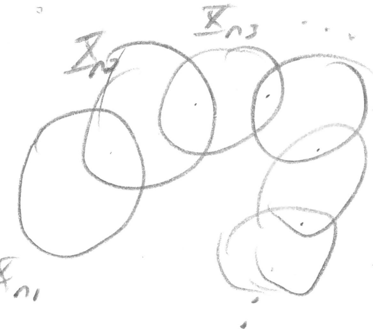
$\boxed{\text{Corollary: } \mathbb{R}^n \text{ is connected}}$

Pf: $\mathbb{R}^n$ is the union of all lines through the origin. None of these are disjoint since they all cross at $(0,0)$ and they are all Homeomorphic to $\mathbb{R}$ and thus connected. $\blacksquare$

$\boxed{\text{Thm: If each pair of pts } x, y \in X \text{ lies in some connected subset } E_{xy} \text{ of } X, \text{ then } X \text{ is connected.}}$

Pf: Fix $a \in X$. Then $X = \bigcup E_{ax}$ and $a \in \bigcap E_{ax} \neq \emptyset$, and each E_{ax} is connected by hypothesis, so X is connected. $\blacksquare$

$\boxed{\text{Thm: If } X = \bigcup_{n=1}^{\infty} X_n \text{ where each } X_n \text{ is connected and } X_{n-1} \cap X_n \neq \emptyset \; \forall n \geq 2, \text{ then } X \text{ is connected.}}$



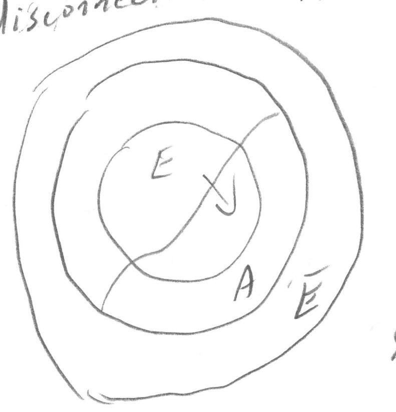
Pf: X_1 is connected. Suppose for some $i=1,\dots,n-1$ we have that $\bigcup_{i=1}^{n-1} X_i$ is connected. Then by Hypothesis $X_n \cap X_{n-1} \neq \emptyset \Rightarrow X_n \cap \bigcup_{i=1}^{n-1} X_i \neq \emptyset$

$\Rightarrow \bigcup_{i=1}^{n} X_i$ is connected since X_n is connected. ∎

Thm: If $E \subseteq X$ is connected and $E \subseteq A \subseteq \bar{E}$, then A is connected

Pf: Let E be connected, $E \subseteq A \subseteq \bar{E}$ suppose $A = C \cup D$ is a disconnection of A. Then by prvs facts, $E \subseteq C$ or $E \subseteq D$.

WLOG suppose $E \subseteq C$. Then $\bar{E} \subseteq \bar{C}$. Since C and D are mutually separated, $\bar{C} \cap D = \emptyset$. But $A \subseteq \bar{E} \subseteq \bar{C} \Rightarrow A \cap D = \emptyset \Rightarrow D = \emptyset$. So C and D can't be a disconnection, contradiction. ∎



Thm: A nonempty product space is connected iff each factor space is connected

$\Rightarrow$ If $X = \prod X_\alpha$ is connected, then since $\pi_\beta : X \to X_\beta$ is continuous, $\pi_\beta(X) = X_\beta$ is connected for each β.

$\Leftarrow$ Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$ be a set of connected spaces.

Let $a \in X = \prod X_\alpha$. Let $E = \{x \in X : \exists E_{ax} \subseteq X$ st E_{ax} is connected, $a, x \in E_{ax}\}$.

By a previous fact, E is connected. If we can show that $\overline{E} = \mathbb{X}$, then $E \subseteq \mathbb{X} \subseteq \overline{E}$ so by the prev thm we'll have $\mathbb{X}$ is connected. Let $U = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$ be a basic open set in $\mathbb{X}$. Pick $b_{\alpha_i} \in U_{\alpha_i}$ for each $i=1,\dots,n$

Define $E_1 = \{ c \in \mathbb{X} : c_{\alpha_1}$ is anything, $c_\alpha = a_\alpha$ otherwise$\}$

$E_2 = \{ c \in \mathbb{X} : c_{\alpha_1} = b_{\alpha_1}, c_{\alpha_2}$ is anything, $c_\alpha = a_\alpha$ otherwise$\}$

$\vdots$

$E_n = \{ c \in \mathbb{X} : c_{\alpha_i} = b_{\alpha_i}$ for each $i=1,\dots,n-1$, c_{α_n} arbitrary, $c_\alpha = a_\alpha$ else$\}$

Then E_k is homeomorphic to $\mathbb{X}_{\alpha_k}$ for each $k=1,\dots,n$, and thus connected. Also, $E_k \cap E_{k+1} \neq \emptyset$ by construction, so $\bigcup_{k=1}^{n} E_k = F$ is connected. Note that $a \in F$, also by construction, and so $F \subseteq E$.

But, by construction, $F \cap U \neq \emptyset$, so $E \cap U \neq \emptyset$, so E is dense in $\mathbb{X}$, ie $\overline{E} = \mathbb{X}$, so we're done. $\blacksquare$

<u>Def</u>: Let $x \in \mathbb{X}$. Then the <u>component</u> of x, C_x, is the largest connected subset of $\mathbb{X}$ which contains x. This is well defined as it is clearly the union of all connected subsets which contain x, and is nonempty, since at the very least, $\{x\}$ is connected.

If $x \neq y$, then either $C_x = C_y$ or $C_x \cap C_y = \emptyset$, since otherwise $C_x \cup C_y$ would be a connected set containing x and y which is strictly larger than either, contradicting their definition.

Thus, $\boxed{\text{The connected components of a space form a partition}}$

<u>Defⁿ</u>: A space is <u>path connected</u> iff for any two points x and y in X, $\exists$ a cts function $f:[0,1]\to X$ st $f(0)=x$, $f(1)=y$. This function is called a <u>path</u> from x to y. (The image $f([0,1])$ is sometimes also referred to as a path).

Wait, back up to components a sec:

Thm: The components of a space X are closed.

<u>Pf</u>: If C is the component containing x, then $C \subseteq \bar{C} \subseteq \bar{C}$
so $\bar{C}$ is a connected set containing x and so since C is maximal,
$\bar{C} \subseteq C$, ie $\bar{C} = C$, ie C is closed. ∎

<u>Defⁿ</u> A space is <u>arc-connected</u> iff for any $x, y \in X$, $\exists$ a path from x to y which is a homeomorphism.

Thm: arc-connected $\Rightarrow$ path connected $\Rightarrow$ connected

<u>Pf</u>: The first implication is trivial. If H and K disconnect X but X is path connected, then for $x \in H$, $y \in K$, $\exists$ a cts function $f: I \to X$ st $f(0)=x$, $f(1)=y$. But clearly $f^{-1}(H) \cap f^{-1}(K) = \emptyset$, and $f^{-1}(H) \cup f^{-1}(K) = f^{-1}(H \cup K) = f^{-1}(X) = I$, and H, K open $\Rightarrow$ $f^{-1}(H)$ and $f^{-1}(K)$ are open, so $f^{-1}(H)$ and $f^{-1}(K)$ disconnect I, $\divideontimes$. ∎

Counterexample: connected $\not\Rightarrow$ path connected.

Let $S = \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1)$ Note that since the identity map is continous as is $g:(0,1] \to \mathbb{R}_{std}$ defined by $x \mapsto \sin(\frac{1}{x})$, The function $g:(0,1] \to \mathbb{R}^2_{std}$ by $x \mapsto (x, \sin(\frac{1}{x}))$ is cts in the product space $\mathbb{R}^2$, since $\pi_1(f(x)) =$ the identity, $\pi_2(f(x)) = y$. since $(0,1]$ is connected and the continuous image of a connected set is connected, S is connected, and furthermore $\bar{S}$ is connected.

$\bar{S}$ is called the topologist's sine curve. It can be shown that $\bar{S} = S \cup \{0\} \times [-1,1]$.

Lemma: (Intermediate Value Theorem): Let $f : X \to Y$ be cts, X connected, Y an ordered set with the order topology. If $a < b \in X$, $r \in Y$ is st r is between $f(a)$ and $f(b)$, then $\exists c \in X$ st $f(c) = r$.

Pf: The sets $A = f(X) \cap (-\infty, r)$, $B = f(X) \cap (r, \infty)$ are clearly disjoint and nonempty, since $f(a)$ is in one while $f(b)$ is in the other. Each is open in $f(X)$. X is connected, and there doesn't exist a $c \in X$ st $f(c) = r$. Then $A \cup B = f(X)$ so then A and B are a disconnection of $f(X)$, # since f is cts. $\boxed{\frac{1}{2}}$

so, suppose $\bar{S}$ is path connected. Then $\exists f : [0,1] \to \bar{S}$ st $f(0) = (0,0)$ and $f(1) = (x, \sin(\frac{1}{x}))$ for some fixed $x \in (0,1]$. Since $\{0\} \times [-1,1]$ is closed, $f^{-1}(\{0\} \times [-1,1])$ is also closed, so it has a maximal element, call it b. Then $f|_{[b,1]}$ is a cts function, and since $[b,1]$ is homeomorphic to $[0,1]$, it corresponds to a path $f : [0,1] \to \bar{S}$ st $f(0) = b$, $f(1) = (x, \sin(\frac{1}{x}))$, and $f(0)$ is the only point which is not in S.

Let $f(t) = (x(t), y(t))$. $x(0) = 0$, while $x(t) > 0$, and $y(t) = \sin(1/x(t))$ $\forall t > 0$. Let $n \in \mathbb{N}$. Then $x(\frac{1}{n}) > 0$.

Note that for any $x(\frac{1}{n})$, $\exists m \in \mathbb{N}$ st $\frac{1}{m} < x(\frac{1}{n})$, ie $\frac{1}{\frac{\pi}{2} + 2\pi m} \in (0, x(\frac{1}{n}))$ and $\sin\left(\frac{1}{\frac{1}{\frac{\pi}{2}+2\pi m}}\right) = 1$. Similarly. This combined with an identical argument gives that $\forall n \in \mathbb{N}$, $\exists u \in (0, x(\frac{1}{n}))$ st $\sin(\frac{1}{u}) = (-1)^n$. Since $x(t)$ is a cts function on $[0,1]$, $x\restriction_{(0,\frac{1}{n})}$ is cts $\forall n$. $(0, \frac{1}{n})$ is connected, and $(0, x(\frac{1}{n}))$ is an ordered space w/ the standard order topology, so by IVT, $\exists$ $t_n \in (0, \frac{1}{n})$ st $x(t_n) = u$. This defines a sequence $\{t_n\}_{n \in \mathbb{N}}$ st $t_n \to 0$ as $n \to \infty$ and $y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = \sin\left(\frac{1}{u}\right) = (-1)^n$.

ie $y(t_n)$ does not converge, a contradiction since $y(t)$ is cts. Thus $\bar{S}$ is connected but not path connected. $\boxed{\Sigma}$

<u>Def^n</u>: A space X is <u>Locally Connected</u> if each pt $x \in X$ has a nbhd base $\mathcal{B}_x$ consisting of connected sets. Equivalently, $\forall x \in X$, $\forall U \in \tau$ w/ $x \in U$, $\exists$ a connected $V \in \tau$ st $x \in V \subseteq U$.

A space X is <u>locally path connected</u> if each pt $x \in X$ has a nbhd base B_x of path connected sets. Equivalently, $\forall x \in X$, $\forall U \in \tau$ w/ $x \in U$, $\exists$ a path connected $V \in \tau$ st $x \in V \subseteq U$.

<u>Thm</u>: Connected + Locally Path Connected $\Rightarrow$ Path Connected

Example is Murky

Pf: Let $a \in X$, $H = \{x \in X : \exists$ a path from x to $a\}$. Obviously $a \in H$, since The constant function $f(x) = a$ is cts. so $H \neq \emptyset$. If we suppose X is connected, then if it turns out H is clopen, then $H = X$ and we'll be done. To show H is open, let $b \in H$, then since H is locally path connected, $\exists U$ nbhd of b st U is path connected. Thus any $z \in U$ can be joined to b via a path, ie $\exists g : [0,1] \to X$ (cts) st $g(0) = b$, $g(1) = z$. Since $b \in H$, $\exists f : [0,1] \to X$ cts st $f(0) = a$, $f(1) = b$. Let $h : [0,1] \to X$ be defined by

$$h(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

By the pasting lemma h is cts. Further, $h(0) = f(0) = a$, $h(1) = g(1) = z$, so h is a path from a to z, ie $z \in H$ ie $U \subseteq H$. So H is open. To show H is closed, let $b \in \overline{H}$, then again since locally path connected, $\exists U \in \mathcal{T}$ st $b \in U$. Then $U \cap H \neq \emptyset$. Say $z \in U \cap H$. Then z can be joined to b by a path since $b, z \in U$, and a and z can be joined by a path since $a, z \in H$. Thus by the same argument from before, a and b can be joined by a path, ie $b \in H$, ie $\overline{H} \subseteq H$, ie $H = \overline{H}$, ie H is closed. $\boxed{/}$

$\boxed{\text{Locally connected} \not\Rightarrow \text{Connected!}}$ since for example $[0,1) \cup (1,2]$ is locally connected but not connected. (both are trivial)

$\boxed{\text{Connected} \not\Rightarrow \text{Locally Connected!}}$

Thm: A space X is locally connected iff $\forall U \in \tau$, each component of U is open in X.

Pf: Suppose X is locally connected, $U \in \tau$, with C being a component of U. Let $x \in C$. Then since locally connected, $\exists V \in \tau$ st V is connected, $x \in V \subseteq U \subseteq C$, so C is open. Conversely, suppose each component of each open set U is open. Let $x \in X$ arbitrary, $U \in \tau$ a nbhd of X. Let C be the component of U containing x. Then C is open, so X is locally connected. $\blacksquare$

Corollary: The components of a locally connected space are clopen.

Pf: We already knew they were closed. Local connectedness now gives that they are open. $\blacksquare$

Corollary: A compact, locally connected space has a finite number of components.

Pf: If a space is locally connected, then the components of X form an open cover of X, so there is a finite subcover. However, each are disjoint, so removing any component from the set takes away their property of being a cover. Thus we must conclude that the original set was finite. $\blacksquare$

Thm: Every quotient of a locally connected space is locally connected.

Pf: