Compactness, Mostly

Def^n: A <u>cover</u> of a space X is a collection $\mathcal{A}$ of subsets of X st $\bigcup\mathcal{A} \supseteq X$. A cover consisting of open sets is called an <u>open cover</u>. A subcollection of $\mathcal{A}$ which still covers X is called a <u>subcover</u>. A space X is <u>Lindelöf</u> if every open cover of X has a countable subcover.

Thm: 2nd Countable $\Rightarrow$ Lindelöf

<u>Pf</u>: Let $\mathcal{B} = \{B_n\}_{n\in\omega}$ be a ctble base for X. Suppose $\mathcal{U}$ is an open cover of X. Then for each $U \in \mathcal{U}$, and $x \in U$, $\exists B_{x,U} \in \mathcal{B}$ st $x \in B_{x,U} \subseteq U$. Note that since $\mathcal{B}$ is ctble, $\mathcal{B}' = \{B_{x,U} : x \in U, U \in \mathcal{U}\} \subseteq \mathcal{B}$ is also ctble. Clearly, $\forall x \in X$, $x \in U$ for some $U \in \mathcal{U}$, so $x \in B_{x,U}$ for some $B_{x,U} \in \mathcal{B}'$. Thus $X = \bigcup\mathcal{B}'$, ie $\mathcal{B}'$ is a cover of X. Since $\mathcal{B}'$ is ctble, $\mathcal{B}' = \{B_{x_n,U_n}\}_{n\in\omega}$, but then since $B_{x_n,U_n} \subseteq U_n \in \mathcal{U}$, $X = \bigcup_{n\in\omega} B_{x_n,U_n} \subseteq \bigcup_{n\in\omega} U_n$, so U_n is a ctble subcover. $\square$

Thm: For (X, ρ) a metric space, 2nd ctble $\Leftrightarrow$ Lindelöf

<u>Pf</u>: Suppose X is Lindelöf. Let $\mathcal{U}_n = \{B_\rho(x, \frac{1}{n}) : x \in X\}$. Clearly, for each $n \in \omega$, $\mathcal{U}_n$ is an open cover of X, so it has a ctble subcover $\mathcal{U}_n^*$. Let $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{U}_n^*$. This is a ctble union of countable sets, so $\mathcal{U}$ is countable. Further, let $W \in \tau_\rho$ with $x \in W$, then of course, $\exists m \in \omega$ st $B_\rho(x, \frac{1}{m}) \subseteq W$, since $\mathcal{B}_x = \{B_\rho(x, \varepsilon) : \varepsilon > 0\}$ is a nbhd base and the archimedean property. Now, since $\mathcal{U}_{2m}^*$ covers X, $\exists B_\rho(y, \frac{1}{2m}) \in \mathcal{U}_{2m}^*$ st $x \in B_\rho(y, \frac{1}{2m})$. But then $\forall z \in B_\rho(y, \frac{1}{2m})$, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

$$< \frac{2}{2m} = \frac{1}{m},$$

So $z \in B_\rho(x, \frac{1}{m})$. Thus, $x \in B_\rho(y, \frac{1}{2m}) \subseteq B_\rho(x, \frac{1}{m}) \subseteq W$

So $\mathcal{U}$ is a base for (X, ρ). Thus X is 2^{nd} ctble. $\boxtimes$

Thus, we have

$$\begin{array}{ccccc} & & 2^{nd} \text{ ctble} & & \\ & \swarrow & \Downarrow & \searrow & \\ 1^{st} \text{ ctble} & & \text{Seperable} & & \text{Lindelöf} \end{array}$$

Where in the case of metric spaces, 2^{nd} ctble, Seperable, and Lindelöf are all equivalent, and 1^{st} ctble is a given.

Thm: If a space X is regular and Lindelöf, then it is normal
[Thus, T_3 + Lindelöf $\Rightarrow T_4$]

Pf: (Directly) Let A, B be disjoint, closed sets in X. For each $a \in A$, Let U_a be an open nbhd of a st $\overline{U_a} \cap B = \emptyset$ (can do since $a \in B^c \in \tau \Rightarrow \exists$ a shrinking U_a st $a \in U_a \subseteq \overline{U_a} \subseteq B^c \Rightarrow \overline{U_a} \cap B = \emptyset$)
Similarly, for each $b \in B$, Let V_b be an open nbhd st $\overline{V_b} \cap A = \emptyset$

Lemma: Closed subspaces of Lindelöf spaces are Lindelöf (not closed, all bets off)

Pf: If $F \in \lambda$, X Lindelöf, then $\{U_\alpha\}_{\alpha \in I}$ a rel. open cover of F
$\Rightarrow \exists \{V_\alpha\}_{\alpha \in I}$ st $V_\alpha \in \tau$ $\forall \alpha$ and $U_\alpha = V_\alpha \cap F$. Then the collection $\{V_\alpha\} \cup \{X - F\}$ is an open cover of X. Since X is Lindelöf, so there is a ctble subcover $\{V_{\alpha_n}\} \cup \{X - F\}$. But then the collection $\{U_{\alpha_n}\}_{n \in \mathbb{N}}$ is a ctble subcover of F, since $F \cap (X - F) = \emptyset$
$\Rightarrow F \subseteq \bigcup V_{\alpha_n} \Rightarrow F \subseteq (\bigcup V_{\alpha_n}) \cap F = \bigcup V_{\alpha_n} \cap F = \bigcup U_{\alpha_n}$. So the subspace F is Lindelöf. $\boxtimes$

Getting back to the thm, since A, B are closed subspaces of Lindelöf space, they are Lindelöf. So .. $\frown$

/... since then $\exists$ a countable cover $\mathcal{U}_z$ $\forall z \in \mathbb{Z}$, and then $\bigcup_{z\in\mathbb{Z}} \mathcal{U}_z$ will be a countable cover of all of E. Suppose bwoc that $\sup C_a = b \in \mathbb{R}$ for some b. (by ctbly covered, we mean a countable subcover of $\mathcal{U}$, the one we fixed). Let $b_n = \frac{b-a}{2^n}$, for $n \in \omega$. Then $\forall n \in \omega$, $a < b_n < b$, with $b_n \to b$ as $n \to \infty$. Since b is the sup, $\forall n \in \omega$, we have a countable collection $\mathcal{A}_n \subseteq \mathcal{U}$ st $[a, b_n]$ is covered by $\mathcal{A}_n$. Note then that $[a, b)$ is covered by $\bigcup \mathcal{A}_n$, a countable subcover. Since $\mathcal{U}$ is a cover of E, $\exists U \in \mathcal{U}$ st $b \in U$. Then $\exists [b, b+\varepsilon) \subseteq \mathbb{R}$ st $[b, b+\varepsilon) \subseteq U$. Then the countable collection $\bigcup \mathcal{A}_n \cup \{U\}$ covers $[a, b+\varepsilon)$, which is a contradiction since $b = \sup C_a \Rightarrow b \geq b+\varepsilon$. $\boxtimes$

Corollary: $E \times E$ is not Lindelöf (And thus we have that a product of Lindelöf spaces need not be Lindelöf)

Pf: We proved earlier that E was T_4 but $E \times E$ was not T_4. However, $T_4 \Rightarrow T_{3\frac{1}{2}}$, and a product of $T_{3\frac{1}{2}}$ spaces is $T_{3\frac{1}{2}}$, so $E \times E$ is $T_{3\frac{1}{2}}$ and thus regular. Thus, to suppose $E \times E$ is Lindelöf would be a contradiction, since regular + Lindelöf $\Rightarrow$ normal, and $E \times E$ is not normal. $\boxtimes$

Some Counterexamples:

Seperable $\not\Rightarrow$ Lindelöf: $E \times E$ works, since a finite product of seperable spaces is seperable, so $E \times E$ is seperable, but we just showed it's not Lindelöf.

... so since the collections $\{U_a : a\in A\}$ and $\{V_b : b\in B\}$ are open covers of A and B, $\exists$ ctble subcovers $\{U_n\}$ and $\{V_n\}$.

Now, define $S_1 = U_1$, $T_1 = V_1 - \overline{S_1}$, $S_2 = U_2 - \overline{T_1}$, $T_2 = V_2 - \overline{(S_1 \cup S_2)}$, $S_3 = U_3 - \overline{(T_1 \cup T_2)}$, $T_3 = V_3 - \overline{(S_1 \cup S_2 \cup S_3)}$, ...

Let $S = \bigcup S_n$, $T = \bigcup T_n$. Clearly, S_n and T_n are open $\forall n\in\mathbb{N}$, so S and T are open. If $a\in A$, then $a\in U_n$ for some $n\in\mathbb{N}$.

Furthermore since $\overline{V_n}\cap A = \emptyset$, $a\notin \overline{V_m}$ for any m, so in particular fixing $m = n$, we have $a\notin \overline{T_n}$, since $T_n = V_n - \overline{\bigcup_{i=1}^{n} S_i}$

$= \overline{V_n \cap \left(\bigcup_{i=1}^{n} S_i\right)} \subseteq \overline{V_n} \cap \left(\overline{\bigcup_{i=1}^{n} S_n}\right) \subseteq \overline{V_n}$, we have $a\notin \overline{T_n}$

$\Rightarrow a\notin \overline{T_1}\cup\cdots\cup\overline{T_n} = \overline{T_1\cup\cdots\cup T_n} \Rightarrow a\in U_n - \overline{\left(\bigcup_{i=1}^{n} T_i\right)} = S_n$.

Thus, $A\subseteq \bigcup S_n = S$, and $B\subseteq \bigcup T_n = T$. Finally, for any S_n, T_n

$S_n\cap T_n$ is clearly empty, since $S_n \subseteq \bigcup_{i=1}^{n} S_i \subseteq \overline{\bigcup_{i=1}^{n} S_i}$

$\Rightarrow S_n \cap \left(X - \overline{\bigcup_{i=1}^{n} S_i}\right) = \emptyset \Rightarrow S_n \cap \left(V_n - \overline{\bigcup_{i=1}^{n} S_i}\right) = \emptyset$

$\Rightarrow \left(\bigcup S_n\right)\cap\left(\bigcup T_n\right) = \emptyset \Rightarrow S\cap T = \emptyset$. So S and T are disjoint open sets with $A\subseteq S$, $B\subseteq T$, ie X is normal. $\boxed{\cancel{}}$

$\boxed{\underline{\text{Thm}}\text{: } E \text{ is Lindelöf}}$

Pf: Let $\mathcal{U}$ be an open cover of E. Let $a\in\mathbb{R}$, and consider the set $C_a = \{x : x\geq a \text{ and } [a,x] \text{ is countably covered}\}$.

If it can be shown that $\sup C_a = \infty$, then we will be done...

Lindelöf $\not\Rightarrow$ 2nd ctble: Ex $\boxed{\text{A}}$

Lindelöf $\not\Rightarrow$ Seperable : Consider $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. Let $x^* \notin \mathbb{R}$, and consider the topology on $\mathbb{R} \cup \{x^*\}$ defined by the following [garbage, need a better one]

Lindelöf $\not\Rightarrow T_0$ [And subsequently $\not\Rightarrow T_{\text{anything}}$] : $(X, \tau_{\text{trivial}} = \{\emptyset, X\})$ with $|X| > 1$. Any open cover is already finite since it would just be X. But This space is not T_0. $\boxed{\text{A}}$

Subspace of a Lindelöf Space which is not Lindelöf : Consider $[0, \omega_1]$. This space is Lindelöf, since if $\mathcal{U}$ covers it, $\omega_1 \in U$ for some open U, ie $\exists\, \gamma < \omega_1$ st $(\gamma, \omega_1] \subseteq U$ but then $[0, \gamma] = \gamma + 1 < \omega_1$ is a countable set, so just pick all that remains and this is a countable set, so just pick $U_\alpha \in \mathcal{U}$ for each $\alpha \in [0, \gamma]$. However, $[0, \omega_1) \subseteq [0, \omega_1]$ is not Lindelöf: If $\alpha \in [0, \omega_1)$, set $U_\alpha = [0, \alpha)$. Then clearly $\mathcal{U} = \{[0, \alpha) : \alpha \in [0, \omega_1)\}$ is an open cover of $[0, \omega_1)$. Suppose $\{U_{\alpha_n}\}_{n \in \omega}$ were a countable subcover of $\mathcal{U}$. Then $\sup\{\alpha_n\}_{n \in \omega} = \bigcup_{n \in \omega} \alpha_n = \omega_1$, a contradiction since then ω_1 is countable. $\boxed{\text{A}}$

1st ctble $\not\Rightarrow$ Lindelöf : $\mathbb{R}_{\text{discrete}}$ is first ctble, but $\{\{x\}\}_{x \in \mathbb{R}}$ is a cover which clearly has no countable subcover. $\boxed{\text{A}}$

A space X is <u>compact</u> iff every open cover of X has a finite subcover. Clearly any compact space is also Lindelöf.

A space is <u>countably compact</u> iff every countable open cover has a finite subcover. Thus, $\boxed{\text{countably compact} + \text{Lindelöf} \iff \text{Compact.}}$

$\boxed{\text{Thm: } [0,1]_{std} \text{ is compact.}}$

<u>Pf</u>: Let $\mathcal{U}$ an open cover. Let $K = \{c \in [0,1] : [0,c] \text{ has a finite subcover}\}$. Clearly, $0 \in K$, so $K \neq \emptyset$. Also, if $c \in K$ and $b \leq c$, then $b \in K$. Pair this with the fact that $0 \in K \Rightarrow \exists U \in \mathcal{U}$ st $0 \in U$ $\Rightarrow \exists (-\varepsilon, \varepsilon) \subseteq \mathbb{R}$ st $0 \in (-\varepsilon, \varepsilon) \subseteq U \Rightarrow [0,b) \subseteq K$ for some $0 < b \in \mathbb{R}$. ie, K is a subinterval of $[0,1]$ which contains 0. Note further that K is rel. open in $[0,1]_{std}$, since $b \in K \Rightarrow [0, b+\varepsilon) \subseteq K$ for some $\varepsilon > 0$ and $[0, b+\varepsilon) = (-\varepsilon, b+\varepsilon) \cap [0,1]$. If $k = \sup K$, then since $\mathcal{U}$ is a cover, $\exists U \in \mathcal{U}$ $k \in U$, $\Rightarrow \exists (k-\varepsilon, k] \subseteq U$ since U is rel. open. Since $k - \varepsilon < k$, $[0, k-\varepsilon]$ has a finite subcover $\{U_i\}_{i=1}^{n}$ so then $\{U_i\}_{i=1}^{n} \cup \{U\}$ is a finite subcover of $[0,k]$, so $k \in K$. Thus, $K = [0,k]$ for some $0 < k \leq 1$, but this means K is closed, and thus clopen. The only clopen set in $[0,1]_{std}$ is $[0,1]$ itself. So $K = [0,1]$, ie $[0,1]$ itself has a finite subcover. ☒

$\boxed{\text{Thm: } [0, \omega_1] \text{ is compact}}$

<u>Pf</u>: Let $\mathcal{U}$ be an open cover. Let U_1 be an open set in $\mathcal{U}$ containing ω_1. Then $\exists \alpha < \omega_1$ st $(\alpha, \omega_1] \subseteq U_1$. Let

$\therefore$ Let $\alpha_1 = \min\{\alpha < \omega_1 : (\alpha, \omega_1] \subseteq U_1\}$. If $\alpha_1 = 0$, then we're done, since then let U_2 be an open set containing 0, and then $\{U_1, U_2\}$ is a finite subcover. Else, let Let $\alpha_2 = \min\{\alpha < \alpha_1 : (\alpha, \alpha_1] \subseteq U_2\}$. Continuing like this, the process must terminate, ie $\exists n \in \omega$ st $\alpha_n = 0$, since otherwise $\{\alpha_n\}$ would be an infinite decreasing chain, contradicting ω_1 being a well ordered set. Then the set $\{U_i\}_{i=1}^{n}$ is a finite subcover, so $[0, \omega_1]$ is compact. $\boxed{\checkmark}$

Thm: $[0, \omega_1)$ is countably compact but not compact (recall also not Lindelöf)

Pf: Let $\mathcal{U} = \{U_n\}_{n\in\omega}$ be a countable open cover of $[0, \omega_1)$. Suppose no finite subcover exists. Then for each $n \in \omega$, $[0, \omega_1) \cap \left(\bigcup_{i=1}^{n} U_i\right)^c$ is nonempty, ie $\exists \alpha_n < \omega_1$ st $\alpha_n \notin \bigcup_{i=1}^{n} U_i$. Let $\alpha = \sup\{\alpha_n\}_{n\in\omega}$. Clearly $\alpha < \omega_1$ since no sequence of countable ordinals can converge to ω_1. Note then that the subspace $[0, \alpha]$ has no finite subcover since otherwise there would be an n st $[0, \alpha] \subseteq \bigcup_{i=1}^{n} U_i$ ie $\alpha \in \bigcup_{i=1}^{n} U_i$, which is impossible. But $[0, \alpha]$ is a closed subspace of $[0, \omega_1]$, which is compact, so it is compact, ie it has a finite subcover, $\#$. So $[0, \omega_1)$ is countably compact. However, it is not compact! Consider the cover $\mathcal{U} = \{[0, \alpha) : \alpha < \omega_1\}$. Clearly this is an open cover of $[0, \omega_1)$, but has no finite subcover (why?)

Thm: If X is compact and $F \subseteq X$ is closed, then the subspace F is compact.

Pf: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of F, ie for each U_α $\exists V_\alpha \in \tau$ st $U_\alpha = V_\alpha \cap F$. Then $F \subseteq \bigcup U_\alpha \subseteq \bigcup V_\alpha$, and thus the collection $\mathcal{U} \cup \{X - F\}$ is an open cover of X. Since X is compact, we have a finite subcover $\{V_{\alpha_i}\}_{i=1}^{n} \cup \{X - F\}$. But then since certainly $F \cap (X - F) = \emptyset$, $F \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$

$$\iff F \subseteq \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cap F = \bigcup_{i=1}^{n} (V_{\alpha_i} \cap F) = \bigcup_{i=1}^{n} U_{\alpha_i},$$

so $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover for F, and thus F is compact. $\boxed{\text{T}}$

It is not in general the case that compact sets are closed! However, in a Hausdorff space it is the case:

Thm: If X is a Hausdorff space and $Y \subseteq X$ is compact, then A is closed.

Pf: Let Y be a compact subspace of a T_2 space X. Let $x_0 \in X - Y$. (If $X - Y = \emptyset$ ie $X = Y$ then Y is already closed). For each $y \in Y$, since X is T_2 $\exists U_y, V_y \in \tau$ st $x_0 \in U_y$, $y \in V_y$, $U_y \cap V_y = \emptyset$. Then the collection $\{U_y : y \in Y\}$ is an open cover of Y, so since Y is compact it has a finite subcover $\{V_{y_i}\}_{i=1}^{n}$. Then $Y \subseteq \bigcup_{i=1}^{n} V_{y_i} := V$. Since $x_0 \in U_{y_i}$ for each i, $x_0 \in \bigcap_{i=1}^{n} U_{y_i} := U$.

Further,
$$V \cap U = (V_{y_1} \cup \dots \cup V_{y_n}) \cap \left(\bigcap_{i=1}^{n} U_{y_i}\right)$$
$$\subseteq (V_{y_1} \cap U_{y_1}) \cup \dots \cup (V_{y_n} \cap U_{y_n}) = \emptyset.$$ Thus,

U is an open nbhd of x_0 st $U \cap V = \emptyset$, but $U \cap Y \subseteq U \cap V = \emptyset$

$\Rightarrow U \cap Y = \emptyset$, so $X - Y$ is open, ie Y is closed. $\boxed{}$

Thm: The continuous image of a compact space is compact

Pf: Let X compact, $f: X \to Y$ cts. If $\mathcal{U}$ is an open cover of Y, then $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X, and thus has a finite subcover $\{f^{-1}(U_i)\}_{i=1}^{n}$. But since we are assuming Y is an image of f, ie $f(X) = Y$, we must have that $f(f^{-1}(U_1)) \cup \dots \cup f(f^{-1}(U_n)) = Y$, but since f is onto $f(f^{-1}(U_i)) \subseteq U_i$, so certainly $\bigcup_{i=1}^{n} U_i = Y$, ie $\{U_i\}_{i=1}^{n}$ is a finite subcover. So Y is compact. $\boxed{}$

Corollary: Any closed interval $[a,b] \subseteq \mathbb{R}_{std}$ is compact.

Pf: Since $[0,1]$ is homeomorphic to $[a,b]$, $\exists$ a cts function $f: [0,1] \to \mathbb{R}$ st $f([0,1]) = [a,b]$ so $[a,b]$ is compact. $\boxed{}$

A subset $A \subseteq \mathbb{R}_{std}$ is <u>bounded</u> if $\exists M \in \mathbb{R}$ st $\forall x \in A$, $|x| \leq M$.

Thm: If $A \subseteq \mathbb{R}_{std}$ is compact then it is bounded.

Pf: Note that the collection $\mathcal{U} = \{B_p(0,n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$, so certainly an open cover of A. Thus, A compact $\Rightarrow \exists M \in \mathbb{N}$ st $A \subseteq \bigcup_{n=1}^{M} B_p(0,n) = B_p(0,M)$, ie $\forall x \in A$, $|x - 0| \leq M$ ie $|x| \leq M$, so A is bounded. $\boxed{}$

Note A is bounded iff $\exists a,b \in \mathbb{R}$ st $A \subseteq [a,b]$.

Corollary: A subset A in $\mathbb{R}_{std}$ is compact iff it is closed and bounded

Pf: $\Rightarrow$ if $A \subseteq \mathbb{R}_{std}$ is compact then since $\mathbb{R}_{std}$ is T_2, A is closed. Further, we just showed it was bounded.
Conversely, we showed that any set of the form $[a,b]$ is compact. Any closed, bounded subset is contained in an interval $[a,b]$ for some $a,b \in \mathbb{R}$ and we showed that a closed subset of a compact set is compact. So any closed, bdd set is compact. $\boxed{\text{//}}$

Thm: A compact T_2 space is a T_4 space.

Pf: If X is compact, then it's Lindelöf. Thus, since Lindelöf + Regular $\Rightarrow$ normal, showing X is T_4 amounts to showing it's regular. Let $A \in \hat{\tau}$, $x \notin A$. Since T_2, we can pick, for each $a \in A$, open disjoint sets U_a, V_a st $x \in U_a$, $a \in V_a$. Then the sets $\{V_a : a \in A\}$ cover A. Since A is closed, $A \subseteq X$ w/ X compact, A is compact, so $\exists \{V_{a_i}\}_{i=1}^{n}$ st $A \subseteq \bigcup_{i=1}^{n} V_{a_i} := V$. Then $x \in \bigcap_{i=1}^{n} U_{a_i} := U \in \tau$, and

$$U \cap V \subseteq (V_{a_1} \cap U_{a_1}) \cup \dots \cup (V_{a_n} \cap U_{a_n}) = \emptyset \cap \dots \cap \emptyset = \emptyset.$$

Thus, X is regular. $\boxed{\text{//}}$

Def$^{\underline{n}}$: A collection $\mathcal{S}$ of subsets of X has the finite intersection property iff the intersection of any finite subcollection of $\mathcal{S}$ is nonempty.

Collections w/ this property are, to filters, as subbases are to topologies, in the sense that the collection of all finite intersections is a filter base for some filter.

Thm: TFAE: (1) $\mathbb{X}$ is compact

(2) If $\mathcal{A} \subseteq \lambda$ has the finite intersection property, then $\bigcap \mathcal{A} \neq \emptyset$

(3) Every filter $\mathcal{F}$ on $\mathbb{X}$ has a cluster point

(4) Every net $\{x_\lambda\}_{\lambda \in \Lambda}$ has a cluster point.

(5) Every ultranet in $\mathbb{X}$ converges

(6) Every ultrafilter on $\mathbb{X}$ converges.

Pf: (1) $\Rightarrow$ (2) Suppose $\{E_\alpha\}_{\alpha \in I}$ is a collection of closed sets st $\bigcap_{\alpha \in I} E_\alpha = \emptyset$. Then $\left(\bigcap_{\alpha \in I} E_\alpha\right)^c = \bigcup_{\alpha \in I} E_\alpha^c = \mathbb{X}$, so the collection $\{E_\alpha^c\}_{\alpha \in I}$ is an open cover of $\mathbb{X}$, and thus has a finite subcover $\{E_{\alpha_i}^c\}_{i=1}^n$ ie $\bigcup_{i=1}^n E_{\alpha_i}^c = \mathbb{X} \Rightarrow \bigcap_{i=1}^n E_{\alpha_i} = \emptyset$. so The collection $\{E_\alpha\}_{\alpha \in I}$ does not have the finite intersection property.

(2) $\Rightarrow$ (3) Let $\mathcal{F}$ be a filter on $\mathbb{X}$. Then $\{\overline{F} : F \in \mathcal{F}\}$ is a family of closed sets with the finite intersection property, so (6) $\Rightarrow \bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$, ie $\exists x \in \mathbb{X}$ st $x \in \overline{F}$ $\forall F \in \mathcal{F}$. We claim x is a cluster point of $\mathcal{F}$, ie $\forall U \in \tau$ st $x \in U$, $U^c \notin \mathcal{F}$. If $x \in U$, then of course $x \notin U^c = \overline{U^c}$ since U^c is closed, so $U^c \notin F$ since $x \in \overline{F}$ $\forall F \in \mathcal{F}$. So x is a cluster pt of $\mathcal{F}$.

(3) $\Rightarrow$ (6) Let $\mathcal{U}$ be an ultrafilter on X. Then by (3), $\mathcal{U}$ has a cluster pt. x. Ie, for all open sets U st $x \in U$, U is positive, ie $U^c \notin \mathcal{U}$. But then since $\mathcal{U}$ is an ultrafilter, $U \in \mathcal{U}$, so $\mathcal{U} \to x$. Thus every ultrafilter on X converges.

(6) $\Rightarrow$ (1) Let $\mathcal{U}$ be an open cover, suppose it has no finite subcover. Then for any finite subcollection $\{U_i\}_{i=1}^n$, $X - (\bigcup U_i) \neq \emptyset$. Then note that for any other subcollection $\{V_j\}_{j=1}^m$, $(X - \bigcup(U_i)) \cap (X - \bigcup V_j) = X - \bigcup_{k=1}^{n+m} W_k \neq \emptyset$, so the collection $\{X - \bigcup_{i=1}^n U_i : \{U_i\}_{i=1}^n$ is a collection of open sets$\}$ is a filter base for some filter on X. This filter is contained in some ultrafilter $\mathcal{U}$, so by hypothesis $\exists\, x \in X$ st $\mathcal{U} \to x$ ie if U is open and $x \in U$, then $U \in \mathcal{U}$. But by design, $X - U \in \mathcal{U}$, so $\mathcal{U}$ is no longer a filter, $\rightarrow\leftarrow$. So for some finite subcollection $\{U_i\}_{i=1}^n$, $X \subseteq \bigcup U_i$, ie X is compact.

(4) and (5) are equivalent to (3) and (6) by translation properties between nets and filters. $\square$

$\boxed{\text{Lemma: Let } \{(X_\alpha, \tau_\alpha)\}_{\alpha \in I} \text{ a collection of spaces. Then a filter } \mathcal{F} \to x_0 \in \prod X_\alpha \text{ iff } \forall \alpha \in I,\ \pi_\alpha(\mathcal{F}) \to \pi_\alpha(x_0).}$

Pf: If $\mathcal{F} \to x_0$ in $\prod X_\alpha$, then since each π_α is continuous, $\pi_\alpha(\mathcal{F}) \to \pi_\alpha(x_0)$ $\forall \alpha \in I$. Conversely, suppose $\pi_\alpha(\mathcal{F}) \to \pi_\alpha(x_0)$ for each $\alpha \in I$. Let $\bigcap_{k=1}^n \pi_{\alpha_k}^{-1}(U_{\alpha_k})$ be a basic nbhd of x_0.

Ie for each α_k, $\pi_{\alpha_k}(x_0) \in U_{\alpha_k} \in \mathcal{T}_{\alpha_k}$. But since $\pi_{\alpha_k}(\mathcal{F}) \to \pi_{\alpha_k}(x_0)$,

This gives that $U_{\alpha_k} \in \pi_{\alpha_k}(\mathcal{F})$, ie $\pi_{\alpha_k}^{-1}(U_{\alpha_k}) \in \mathcal{F}$. So by properties

of filters, $\bigcap_{k=1}^{n} \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \in \mathcal{F}$. So any basic nbhd of x_0 is in $\mathcal{F}$.

But then if U is an arbitrary nbhd of x_0, $\exists$ a basic nbhd of x_0,

B st $x_0 \in B \subseteq U$, and $B \subseteq \mathcal{F}$ so $U \in \mathcal{F}$. Thus, $\mathcal{F} \to x_0$. ☒

Thm: Any product of compact spaces is compact

Pf: Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ be a collection of compact spaces.

Then $\prod X_\alpha$ is compact iff any ultrafilter $\mathcal{U}$ on $\prod X_\alpha$

converges iff $\pi_\alpha(\mathcal{U})$ converges for all $\alpha \in I$. But since the X_α

are all compact, $\pi_\alpha(\mathcal{U})$ does converge, so we're done ☒

Corollary (Tychonoff Thm): A product space $\prod_{\alpha \in I} X_\alpha$ is compact iff each factor space X_α is compact.

Pf: If $\prod_{\alpha \in I} X_\alpha$ is compact, then since the cts image of a compact space is compact, $\pi_\alpha\left(\prod X_\alpha\right) = X_\alpha$ is compact for each $\alpha \in I$.

The converse was just shown above. ☒

Corollary: A subspace of $\mathbb{R}^n$ is compact iff it is closed and bdd

Corollary: All cubes are compact

Double Corollary: All cubes are T_4 (since cubes are T_2 + compact $\Rightarrow T_4$)

This allows us to reformulate $T_{3\frac{1}{2}}$ in a few ways:

Thm: TFAE: (1) X is Tychonoff ($T_{3\frac{1}{2}}$)

(2) X embeds into a Cube

(3) X is homeomorphic to a subspace of some compact T_2 space.

(4) X is homeomorphic to a subspace of some T_4 space.