More Separation Axioms (T_4 onward)

Def$^{\underline{n}}$: A space is normal iff whenever A, B are disjoint closed sets in $\mathbb{X}$, $\exists$ disjoint open sets $U, V \in \tau$ st $A \subseteq U$, $B \subseteq V$. A space which is normal and T_1 is called T_4.

Jones' Lemma: If a space $\mathbb{X}$ contains a dense set D and a closed set S which has the discrete topology as a subspace of S, and $|S| \geq 2^{|D|}$, then $\mathbb{X}$ is not normal.

Pf: If S is closed and $T \subseteq S$, and S has the discrete top. as a subspace of $\mathbb{X}$, then T is closed rel. to S, ie $\exists$ a closed $F \subseteq \mathbb{X}$ st $T = F \cap S$, but this is an intersection of two closed sets and thus T is closed in $\mathbb{X}$. Further, $S - T \subseteq S$, so $S - T$ is also closed in $\mathbb{X}$. Thus $\exists U(T), V(T) \in \tau_{\mathbb{X}}$ st $T \subseteq U(T)$, $S - T \subseteq V(T)$, and $U(T) \cap V(T) = \emptyset$. Let $T_1, T_2 \subseteq S$. Then if $T_1 - T_2 \neq \emptyset$, then $T_1 \cap (S - T_2) \neq \emptyset$, so since $U(T_1) \supseteq T_1$ and $V(T_2) \supseteq S - T_2$, $U(T_1) \cap V(T_2) \neq \emptyset$, and is open in $\mathbb{X}$, so since D is dense, $U(T_1) \cap V(T_2) \cap D \neq \emptyset$. Note that $U(T_1) \cap V(T_2) \cap D \subseteq U(T_1) \cap D$, but since $U(T_2) \cap V(T_2) = \emptyset$, $U(T_1) \cap V(T_2) \cap D \nsubseteq U(T_2) \cap D$. Thus so $U(T_1) \cap D$ and $U(T_2) \cap D$ are different subsets of D, and so the function $\varphi: T_1 \subseteq S \to U(T_1) \cap D$ is injective, and so $|P(S)| \leq |P(D)| \Leftrightarrow 2^{|S|} \leq 2^{|D|} \Rightarrow |S| < 2^{|D|}$, $\#$. So $\mathbb{X}$ is not normal. $\boxed{\nabla}$

$\boxed{\text{Thm: The Moore Plane } \mathbb{R}^2_+ \text{ is } T_{3\frac{1}{2}} \text{ but not } T_4}$

Pf: To show $\mathbb{R}^2_+$ is $T_{3\frac{1}{2}}$, let $\vec{x} \in \mathbb{R}^2_+$, and V an basic nbhd of $\vec{x}$. So V is either a disk centered at $\vec{x}$ or a disk tangent to $\vec{x}$ if $\vec{x}$ is on the x-axis. Define $f: \mathbb{R}^2_+ \to [0,1]$ by setting $f(\vec{x}) = 0$, $f(\vec{y}) = 1$ $\forall \vec{y} \notin V$, and letting f go to 1 linearly along line segments from $\vec{x}$ to the boundary of V. Then f is cts on $\mathbb{R}^2_+$ with $f(\vec{x}) = 0$ and $f(X - V) = 1$. If F is a closed set not containing $\vec{x}$, then $\exists$ a basic nbhd V of x st $F \subseteq X - V$, so then $1 - f(\vec{z})$ is a cts function st $1 - f(\vec{x}) = 1$ and $1 - f(F) = 0$. So $\mathbb{R}^2_+$ is $T_{3\frac{1}{2}}$. Note that the x-axis as a subspace of $\mathbb{R}^2_+$ has the discrete topology and is clearly closed, with $|x\text{-axis}| = 2^\omega$. But we've already seen that $\mathbb{R}^2_+$ is seperable, so it has a dense set D with $|D| = \omega$. Thus $|x\text{-axis}| = 2^\omega = 2^{|D|}$, so by Jones' Lemma, $\mathbb{R}^2_+$ is not normal. ∎

$\boxed{\text{Thm: Any Metric Space is } T_4}$

Pf: Let (X, ρ) is a metric space, let A, B closed in X, disjoint. For $x \in A$, $A \subseteq B^c \in \tau$, so $\exists \delta_x > 0$ st $B_\rho(x, \delta_x) \subseteq B^c$ ie $B_\rho(x, \delta_x) \cap B = \emptyset$. similarly, for $y \in B$, $\exists \varepsilon_y > 0$ st $B_\rho(y, \varepsilon_x) \cap A = \emptyset$.

Let $U = \bigcup_{x \in A} B_\rho(x, \frac{\delta_x}{3})$, $V = \bigcup_{y \in B} B_\rho(y, \frac{\varepsilon_x}{3})$. Then U and V are open sets st $A \subseteq U$, $B \subseteq V$. wts $U \cap V = \emptyset$. BWOC, let $z \in U \cap V$.

$z \in U \Rightarrow \exists B_\rho(x, \frac{\delta_x}{3})$ st $z \in B_\rho(x, \frac{\delta_x}{3}) \Rightarrow \rho(x, z) < \frac{\delta_x}{3}$, and

$z \in V \Rightarrow \exists B_\rho(y, \frac{\varepsilon_y}{3})$ st $z \in B_\rho(y, \frac{\varepsilon_y}{3}) \Rightarrow \rho(y, z) < \frac{\varepsilon_y}{3}$.

But since ρ is a metric, $\rho(x,y) \leq \rho(x,z) + \rho(z,y) < \frac{\delta_x}{3} + \frac{\varepsilon_y}{3}$

(WLOG letting $\delta_x > \varepsilon_y$) $< \frac{\delta_x}{3} + \frac{\delta_x}{3} = \frac{2\delta_x}{3}$

$< \delta_x$.

Thus $\rho(x,y) < \delta_x \Rightarrow y \in B_\rho(x, \delta_x) \Rightarrow y \notin B$ since supposedly $B_\rho(x, \delta_x) \cap B = \emptyset$. #. So $U \cap V = \emptyset$, ie (X, ρ) is normal. Since (X, ρ) is also T_2, this means all metric spaces are T_4. $\boxtimes$

$\boxed{\text{Thm: The Sorgenfrey Line } E \text{ is } T_4.}$

<u>Pf</u>: Let A, B closed in E, disjoint. Then $A \subseteq B^c$, open in E, so if $x \in A$, $x \in B^c$, so $\exists\, \delta_x > 0$ st $[x, x+\delta_x) \subseteq B^c \Rightarrow [x, x+\delta_x) \cap B = \emptyset$.
Similarly, if $y \in B$, $\exists\, \varepsilon_y > 0$ st $[y, y+\varepsilon_y) \cap A = \emptyset$.

Let $U = \bigcup_{x \in A} [x, x+\delta_x)$, $V = \bigcup_{y \in B} [y, y+\varepsilon_y)$. Then $A \subseteq U$, $B \subseteq V$, and U, V open in E. Suppose $U \cap V \neq \emptyset$. To show this, it suffices to show that $\forall\, [x, x+\delta_x), [y, y+\varepsilon_y)$, that these are disjoint. WLOG let $x < y$. Since $[x, x+\delta_x) \cap B = \emptyset$, $y \in B$, then we must have $y \geq x + \delta_x$, and so $[x, x+\delta_x) \cap [y, y+\varepsilon_y) = \emptyset$.

So $U \cap Y = \emptyset$ and so E is normal. If $x \in \mathbb{R}$, then the intersection $\bigcap_{n \in \omega} [x, x+\frac{1}{n}) = \{x\}$ is an intersection of closed sets in E so is closed, so E is T_1. Thus E is T_4. $\boxtimes$

Thm: $E \times E$ is not T_4

Pf: Consider the anti diagonal $-\Delta = \{(x,-x) : x \in \mathbb{R}\}$.
This is a closed set in $E \times E$, since if $(x,y) \notin -\Delta$,
then either $y > -x$ or $y < -x$. If $y > -x$, then the
open set $[x, x+1) \times [y, y+1)$ contains (x,y) and misses $-\Delta$,
else, $y < -x$, so the set $[x, \frac{x-y}{2}) \times [y, \frac{y-x}{2})$ misses $-\Delta$
and contains (x,y). So in either case $(E \times E) - (-\Delta)$ is open.
Clearly, $|-\Delta| = 2^{\omega}$. Also, E is seperable, and a finite
product of seperable spaces is seperable, so $E \times E$ contains a
dense set D with $|D| = \omega$. Note that if $(x,-x) \in -\Delta$,
then $[x, x+1) \times [-x, x+1)$ is an open set in $E \times E$ with
$[x, x+1) \times [-x, x+1) \cap (-\Delta) = \{(x,-x)\}$ so every singleton set
in $-\Delta$ is relatively open, ie $-\Delta$ has the discrete topology
as a subspace of $E \times E$. But then $|-\Delta| = 2^{\omega} = 2^{|D|}$, so
by Jones' Lemma $E \times E$ cannot be normal, and thus is not T_4. ///

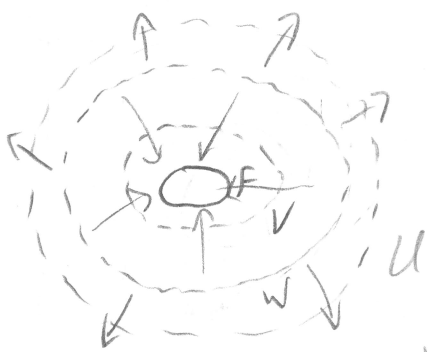
Thus, Thm: A subspace of a T_4 space need not be T_4, and a product of T_4 spaces need not be T_4.

Pf: The product of $E \times E$ proves the product claim.
~~As for subspaces, consider again the moore plane $\mathbb{R}^+_2$.~~
~~As for subspaces define the Tychonoff Plank as the space~~
As for subspaces define the Tychonoff Plank as the space
$[0, \omega_1] \times [0, \omega]$. (Pictured on next page)

Urysohn's Lemma: A space $\mathbb{X}$ is normal iff $\forall$ closed sets A, B in $\mathbb{X}$, with $A \cap B = \emptyset$, $\exists$ a cts function $f: \mathbb{X} \to [0,1]$ st $f(A) = 0$ and $f(B) = 1$.

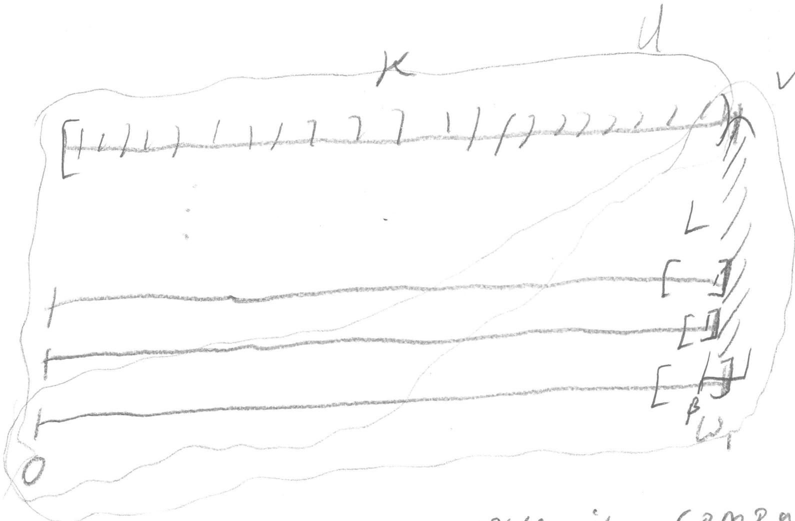
First, a lemma for our lemma: A space is normal iff $\forall$ closed sets F and all open sets U st $F \subseteq U$, $\exists V \in \tau$ st $F \subseteq V \subseteq \bar{V} \subseteq U$. [Ie Normal $\Leftrightarrow$ nbhds of closed sets can be "shrunk" in the same sense that regular $\Leftrightarrow$ nbhds of pts can be "shrunk".]

Pf: $\Rightarrow$) Let F be closed in $\mathbb{X}$. U open w/ $F \subseteq U$. Then $F \cap U^c = \emptyset$, and U^c is closed so these are disjoint closed sets. Thus by normality, $\exists$ open sets $V, W \in \tau$ st $F \subseteq V$, $U^c \subseteq W$, $V \cap W = \emptyset$. Note now though that since



$U^c \subseteq W$, $W^c \subseteq U$, and W^c is closed. Further, $V \cap W = \emptyset \Rightarrow V \subseteq W^c$. So we have the chain $F \subseteq V \subseteq W^c \subseteq U$. Since $\bar{V}$ is the smallest closed set containing V, we thus have $F \subseteq V \subseteq \bar{V} \subseteq U$, as desired.

Conversely, let F, G be closed, disjoint sets in $\mathbb{X}$. Then $F \subseteq G^c$, an open set, so $\exists U$ open st $F \subseteq U \subseteq \bar{U} \subseteq G^c$. $\bar{U} \subseteq G^c \Rightarrow G \subseteq \bar{U}^c$, an open set. Thus, $F \subseteq U$, $G \subseteq \bar{U}^c$, and since $U \subseteq \bar{U}$, $U \cap \bar{U}^c = \emptyset$, so $\mathbb{X}$ is normal. ∎



We will show later that any product of compact spaces is compact, and already know a product of Hausdorff spaces is Hausdorff, and will later show that if a space is compact and Hausdorff, then it's T_4.

Thus, The Tychonoff plank, is T_4. Consider the subspace $[0, \omega_1) \times [0, \omega)$. Consider the sets $K = [0, \omega_1) \times \{\omega\}$, and $L = \{\omega_1\} \times [0, \omega)$. Since $[0, \omega_1] \times \{\omega_1\}$ is closed in the plank, and $K = ([0, \omega_1) \times [0, \omega)) \cap ([0, \omega_1] \times \{\omega_1\})$, K is rel. closed. and $L = ([0, \omega_1) \times [0, \omega)) \cap (\{\omega_1\} \times [0, \omega])$. L is rel. closed.

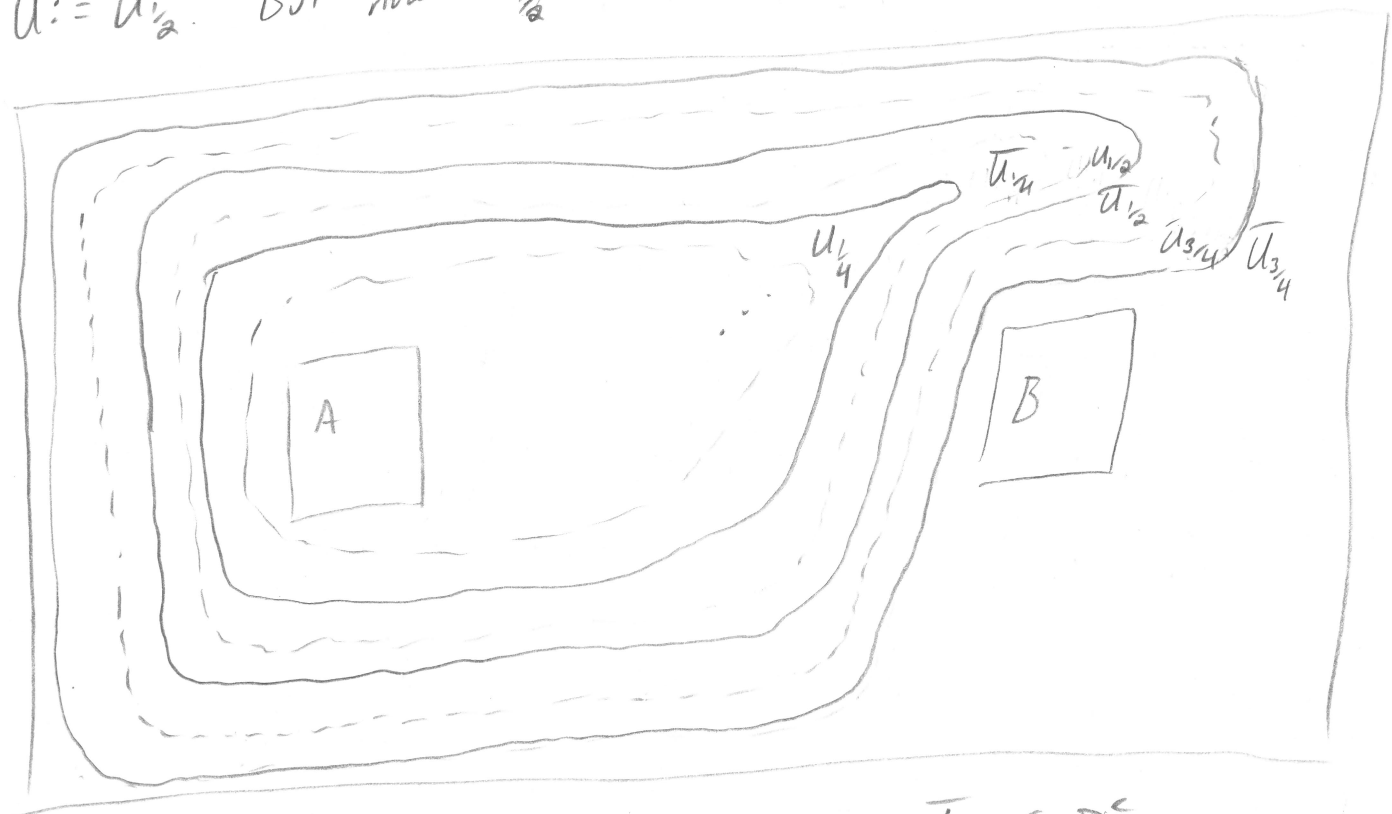
$\nexists$ U, V rel. open st $K \subseteq U$, $L \subseteq V$, $U \cap V = \emptyset$ (clearly K, L are disjoint). Then for each $n \in \omega$, since $\{\omega_1\} \times [0, \omega] \subseteq V$, $(\omega_1, n) \in V$, so $\exists\, \alpha_n \in ON$ st $[\alpha_n, \omega_1] \times \{n\} \subseteq V$.

Let $\beta = \sup\{\alpha_n : n \in \omega\} = \bigcup\{\alpha_n : n \in \omega\}$. $\beta \in ON$, an in particular, $\beta < \omega_1$ since $\alpha_n < \omega_1$ for each α_n, ie it's a ctble union of ctble ordinals. $(\beta, n) \in V$ for each $n \in \omega$, since $\forall n\ \exists\, \alpha_n < \beta$ st $[\alpha_n, \omega_1] \times \{n\} \subseteq V$ and $(\beta, n) \in [\alpha_n, \omega_1] \times \{n\}$. But then the sequence $\{(\beta, n)\}_{n \in \omega} \to (\beta, \omega)$, ie for any rel. open set W containing (β, ω), $\exists N$ st $\forall n \geq N$ $(\beta, N) \in W$. So $\exists N_0$ st $(\beta, N_0) \in U$, but since $(\beta, N_0) \in V$, $U \cap V \neq \emptyset$, so this subspace is not T_4. $\square$

Now, the proof of Urysohn's Lemma: Let X normal, A, B disjoint closed sets in X. Then $\exists$ open sets $U, V \in \tau$ st $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$.

$B \subseteq V \Leftrightarrow B \cap V^c = \emptyset$, and $U \subseteq V^c$ for the same reason.

$A \subseteq U \subseteq \overline{U} \subseteq V^c$ and since $B \cap V^c = \emptyset$, $B \cap \overline{U} = \emptyset$ as well. Let $U := U_{1/2}$. But now $\overline{U}_{1/2}$ and B are disjoint closed sets. Thus...



... $\exists$ an open set $U_{3/4}$ st $\overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq B^c$

Additionally, A and $U_{1/2}{}^c$ are closed, disjoint, so $\exists U_{1/4}$ st

$A \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq (U_{1/2}{}^c)^c = U_{1/2}$ ie...

$A \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq B^c$. Continue in this way to inductively define for all $r \in \{\frac{k}{2^n} : n \in \mathbb{N}, k = 1, 2, \ldots, 2^n - 1\} = D$ sets U_r with the property that for each r,

$A \subseteq U_r$, and $\overline{U}_r \cap B = \emptyset$, and $\overline{U}_r \subseteq U_s$ $\forall r < s$.

Now, define $f: X \to [0,1]$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } \forall r \in D,\ x \notin U_r \\ \inf\{r \in D : x \in U_r\} & \text{otherwise.} \end{cases}$$

Let $x \in A$. Then $x \in U_r\ \forall r$, so $f(x) = 0$. Also, if $x \in B$, then $x \notin U_r\ \forall r \in D$ by construction, so $f(x) = 1$. Thus, we have that $f(A) = 0$, $f(B) = 1$. It remains to show that f is cts.

To do this, first note that if (1) $x \in \overline{U_r}$ for some r, then since $\overline{U_r} \subseteq U_s\ \forall s > r$, we have that $f(x) \leq r$. Also, if (2) $x \notin U_r$, then since $U_s \subseteq U_r\ \forall s < r$, $f(x) \geq r$. Let $x_0 \in X$. Then $\exists (c,d) \subseteq [0,1]$ st $f(x_0) \in (c,d)$. Choose dyadic rationals r, s st $c < r < f(x_0) < s < d$, and consider the set $U_s - \overline{U_r}$. Clearly, this is open. ($U_s - \overline{U_r} = (U_s \cap (X - \overline{U_r}))$, an intersection of open sets) Note that since $f(x_0) < s$, then by the contrapositive of (2), $x_0 \in U_s$, and since $f(x_0) > r$, we have $x \notin \overline{U_r}$ by (1). Thus $x_0 \in U_s - \overline{U_r} = U$. Next, let $x \in U$. Then $x \in U_s \subseteq \overline{U_s}$, so $f(x) \leq q$, by (1). Finally, since $x \notin \overline{U_r}$, $x \notin U_r$, so $f(x) \geq r$. Thus $f(x) \in (r,s) \subseteq (c,d)\ \forall x \in U$ ie $f(U) \subseteq (c,d)$. So f is cts.

Conversely, if A, B disjoint closed with $f: X \to [0,1]$ a cts function with $f(A) = 0$, $f(B) = 1$, then $f^{-1}([0,\frac{1}{2}))$ and $f^{-1}((\frac{1}{2},1])$ will be open sets, disjoint in X, st $A \subseteq f^{-1}([0,\frac{1}{2}))$, $B \subseteq f^{-1}((\frac{1}{2},1])$. So X is normal. $\boxed{\checkmark}$

<u>Corollary</u>: $T_4 \Rightarrow T_{3\frac{1}{2}}$.

<u>Pf</u>: $T_4 \iff T_1$ + Normal. Let $x \in X$, F closed in X with $x \notin F$. Since X is T_1, $\{x\}$ is closed, disjoint from F. So by Normality and Urysohn's Lemma, $\exists$ a cts $f: X \to [0,1]$ st $f(F) = 0$, $f(\{x\}) = 1$, ie $f(x) \neq 0$. Thus, X is $T_{3\frac{1}{2}}$. $\boxtimes$

Note that since $[0,1]$ is homeomorphic to $[a,b]$ for any $a,b \in \mathbb{R}$, and the composition of cts functions is cts, we can always replace the $[0,1]$ in the statement of Urysohn with any interval $[a,b]$.

Also, note that Urysohn in no way implies that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. However, if we have such a function, then by the converse of Urysohn such a space would be T_4.

<u>Def^n</u>: A space is T_5 (or <u>completely normal</u>) if all subspaces of X are T_4. Clearly, $T_5 \Rightarrow T_4$, since X is a subspace of itself.

<u>Def^n</u>: A space is T_6 (or <u>perfectly normal</u>) if there is a Urysohn function $\forall$ closed disjoint A, B, f, st $f^{-1}(0) = A$, $f^{-1}(B) = 1$. ($T_6 \Rightarrow T_5$). Somehow.

<u>Thm (Tietze's Extension Thm)</u>: X is normal iff whenever A is a closed subset of X and $f: A \to \mathbb{R}$ is cts, $\exists$ an extension of f to all of X, ie a cts map $F: X \to \mathbb{R}$ st $F(x) = f(x)$ $\forall x \in A$. (ie $F|_A = f$)

Pf: $\Rightarrow$ Suppose to start that $f: A \to [-1,1]$. Let $A_1 = \{x \in A : f(x) \geq \frac{1}{3}\}$, $B_1 = \{x \in A : f(x) \leq -\frac{1}{3}\}$. is cts!

ie $A_1 = f^{-1}([\frac{1}{3}, 1])$, $B_1 = f^{-1}([-1, -\frac{1}{3}])$. Thus A_1 and B_1 are disjoint closed sets in X. thus, by Urysohn, $\exists$ a cts $f_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ st $f_1(A_1) = \frac{1}{3}$, $f_1(B_1) = -\frac{1}{3}$. Note that $|f(x) - f_1(x)| \leq 1 - \frac{1}{3} = \frac{2}{3}$ $\forall x \in A$. ie $f - f_1 : A \to [-\frac{2}{3}, \frac{2}{3}]$, and is cts. Let $f - f_1 := g_1$.

Let $A_2 = f^{-1}([\frac{2}{9}, \frac{2}{3}))$, $B_2 = f^{-1}((-\frac{2}{3}, -\frac{2}{9}])$, define $f_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ cts st $f_2(A_2) = \frac{2}{9}$, $f_2(B_2) = \frac{-2}{9}$, and note that $|g_1 - f_2| \leq \frac{2}{3} - \frac{2}{9}$
$= \frac{6}{9} - \frac{2}{9} = (\frac{2}{3})^2$,

and so define $g_2 = g_1 - f_2 : A \to [-(\frac{2}{3})^2, (\frac{2}{3})^2]$.

Continue in this way, and note that $\forall n$, $g_n = f - (\sum_{k=1}^{n} f_i)$, with

$$|g_n| = |f - \sum_{k=1}^{n} f_k| \leq \left(\frac{2}{3}\right)^n$$

Define $F(x) := \sum_{i=1}^{\infty} f_i(x)$, and note that F, like all the f_i, is defined $\forall x \in X$ (supposing the series converges)

Note that since $|f_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, $\left|\sum_{i=1}^{\infty} f_i(x)\right| \leq \sum_{i=1}^{\infty} |f_i(x)|$

$$\leq \sum_{i=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1/3}{1-\frac{1}{3}} = 1,$$

so this function is well defined $\forall x \in X$. Next note that $F(x)$ is cts: Let $\varepsilon > 0$. fix $x \in X$. Pick $N \in \mathbb{N}$ st $\sum_{n=N+1}^{\infty} \left(\frac{2}{3}\right)^n < \frac{\varepsilon}{2}$. Each f_i is cts, so for each $i = 1, \ldots, N$, $\exists U_i$ open nbhd of x in X st $y \in U_i \Rightarrow |f_i(x) - f_i(y)| < \frac{\varepsilon}{2N}$.

Let $U = U_1 \cap \ldots \cap U_N$ open nbhd of x, since $N < \infty$. Then note that for $y \in U$, $|F(x) - F(y)| = \left|\sum_{i=1}^{\infty} f_i(x) - \sum_{i=1}^{\infty} f_i(y)\right|$

$$= \left|\sum_{i=1}^{\infty} f_i(x) - f_i(y)\right| \le \sum_{i=1}^{\infty} |f_i(x) - f_i(y)|$$

$$= \sum_{i=1}^{N} |f_i(x) - f_i(y)| + \sum_{i=N+1}^{\infty} |f_i(x) - f_i(y)|$$

$$< \sum_{i=1}^{N} \frac{\varepsilon}{2N} + \sum_{i=N+1}^{\infty} \left(\frac{2}{3}\right)^n$$

since $|f_i(z)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

$\Rightarrow |f_i(x) - f_i(y)| \le \left(\frac{2}{3}\right)^n$

$$< \frac{N\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon$$

ie for any basic open $(-\varepsilon, \varepsilon) \subseteq \mathbb{R}$, $\exists$ an open U in X st $F(U) \subseteq (-\varepsilon, \varepsilon)$. ie F is cts. Next, we show that $F|_A = f$ ie $\forall a \in A$, $F(a) = f(a)$. Consider the n^{th} partial sum $S_n(x) = \sum_{i=1}^{n} f_i(x)$. Note that by construction, $|f(a) - S_n(a)| = |g_n(a)| \le \left(\frac{2}{3}\right)^n$. but this goes to 0 as $n \to \infty$, so we have that $f(a) = \lim_{n\to\infty} S_n(a) = F(a)$.

Finally, note that when we showed that F was well defined, we also showed that it was bounded by 1. Thus, $F: X \to [-1,1]$ is a cts extension of f to all of X. Now we turn to the more general case of $f: X \to \mathbb{R}$. Since $\mathbb{R}$ is homeomorphic to $(-1,1)$, it suffices to show that $f: X \to (-1,1)$ can be extended. Note that there is nothing wrong with regarding f as a map into $[-1,1]$. Thus $\exists$ by the first case a cts extension $F': X \to [-1,1]$.

Let $A_0 = \{x \in X : |F'(x)| = 1\} = (F')^{-1}(1) \cup (F')^{-1}(-1)$. Since F' is cts, this is a union of two closed sets in X and thus closed.

Further, A is closed, disjoint from A_0! So again by Urysohn, $\exists$ a cts $g: X \to [0,1]$ st $g(A)=1$, $g(A_0)=0$. Define $F: X \to (-1,1)$ by $F(x) = g(x)F'(x)$. This maps into $(-1,1)$ since if $F'(x) = \pm 1$, then $g(x) = 0 \Rightarrow F(x) = 0$. Further, if $x \in A$, then $g(x) = 1$, so $F(x) = 1 \cdot F'(x) = f(x)$, so $F|_A = f$. Finally, it's a product of cts functions, so it's cts! This proves the general case.

Conversely, if $A, B \subseteq X$ are closed, disjoint, then $A \cup B$ is closed. Define the function $f: A \cup B \to [0,1]$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. By the pasting lemma and the fact that cts functions are cts, we have by hypothesis a cts function $F: X \to [0,1]$ st $F(A) = 0$ and $F(B) = 1$. This is a Urysohn Function on X! So by the other direction of Urysohn's Lemma, X is normal. $\boxed{/}$

Thm: The closed, cts image of a T_4 space is T_4

Pf: Let X be T_4, f a cts closed function on X. Let $f(X) = Y$. Let A, B closed subsets of Y st $A \cap B = \emptyset$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are closed disjoint sets in X, so X $T_4 \Rightarrow \exists U_1, U_2 \in \mathcal{T}_X$ st $f^{-1}(A) \subseteq U_1$, $f^{-1}(B) \subseteq U_2$, $U_1 \cap U_2 \neq \emptyset$. Since f is a closed map, $f(X - U_1)$ and $f(X - U_2)$ are closed in Y. Thus, $Y - f(X - U_1) = V_1$ and $Y - f(X - U_2) = V_2$ are open in Y. We claim V_1 and V_2 separate A and B.

To show this, let $y \in A$. Then $y = f(x)$ for some $x \in X$.

So $x \in f^{-1}(A) \subseteq U_1 \Rightarrow x \notin X - U_1$

$\Rightarrow f(x) = y \notin f(X - U_1)$

$\Rightarrow y \in Y - f(X - U_1) = V_1$.

So $A \subseteq V_1$, and an identical argument shows $B \subseteq V_2$.

Suppose $V_1 \cap V_2 \neq \emptyset$. Let $y \in V_1 \cap V_2$. Then $y = f(x)$ for some $x \in X$. Then $y \notin f(X - U_1)$ and $y \notin f(X - U_2)$ ie $y \in f(U_1)$ and $y \in f(U_2)$, ie $x \in U_1$ and $x \in U_2$, but this is a contradiction since $U_1 \cap U_2 = \emptyset$. So Y is T_4. $\boxtimes$

$\boxed{\text{A cts open image of a } T_4 \text{ space need not be } T_4.}$

Pf: Consider the line w/ 2 origins as a quotient of $[-1,1] \times \{0,1\} \subseteq \mathbb{R}^2_{std}$. $[-1,1]$ and $\{0,1\}$ are both compact and T_2 subsets of $\mathbb{R}$, so the product is compact and T_2 and thus $X = [-1,1] \times \{0,1\}$ is T_4. Define $\sim$ by $(x,0) \sim (x,1)$ $\forall x \in [-1,1] - \{0\}$. Ie $\forall (x,y) \in X$, $[(x,y)] = \{(x,0),(x,1)\} := x$ $\forall x \neq 0$, and if $x = 0$ then $[(0,0)] = \{(0,0)\} := 0_0$ and $[(0,1)] = \{(0,1)\} = 0_1$.

The quotient map $q : X \to X/\sim$ is then a cts map, and is open :

Let U open in $\mathcal{T}_{\mathbb{X}}$. wts $f(U)$ open in $\mathcal{T}_{\mathbb{X}/\sim}$.

Let $x \in f(U)$. If $x \neq 0_0$ or 0_1, then $x = \{(x,0),(x,1)\}$ or $(x_0, 1)$ for some $x_0 > 0$. Then $\exists\, \varepsilon > 0$ st
$\underset{\text{WLOG}}{}$

$0 < x_0 - \varepsilon < x_0 < x_0 + \varepsilon$, so that $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq U$,

so $f((x_0 - \varepsilon), ($