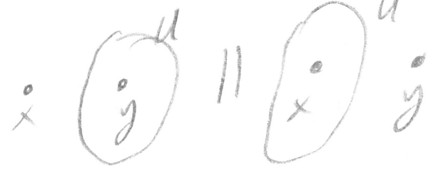
Seperability ($T_0 - T_{3\frac{1}{2}}$)

<u>Def</u>: A space $\mathbb{X}$ is T_0 iff whenever $x, y \in \mathbb{X}$ and $x \neq y$, $\exists\, U \in \tau$ st $x \in U \wedge y \notin U$ or $y \in U \wedge x \notin U$.



Thus points in a T_0 space are topologically distinguishable.

[If $(\mathbb{X}, \tau)$ is the trivial topology and $\exists\, a, b \in \mathbb{X}$ st $a \neq b$, then clearly $\mathbb{X}$ is not T_0.]

Fact: A pseudometric ρ is a metric iff the topology induced by ρ is T_0

Pf: Metric spaces are obviously T_0, since if $x \neq y \in \mathbb{X}$, then let $0 < \varepsilon < \rho(x,y)$, then $x \in B_\rho(x, \varepsilon)$ and $y \notin B_\rho(x, \varepsilon)$. Conversely, if $(\mathbb{X}, \rho)$ is T_0, then $x \neq y$ in $\mathbb{X} \Rightarrow \exists\, U \in \tau$ st (wLOG) $x \in U$, $y \notin U$, so $\exists\, B_\rho(x, \varepsilon)$ st $x \in B_\rho(x, \varepsilon) \subseteq U$, any $y \notin U \Rightarrow y \notin B_\rho(x, \varepsilon)$ $\Rightarrow \rho(x,y) > \varepsilon > 0$. ☒

Fact: A subspace of a T_0 space is T_0, arbitrary products of T_0 spaces are T_0. Arbitrary quotients of T_0 spaces need not be T_0.

Pf: If $A \subseteq \mathbb{X}$, $\mathbb{X}$ T_0, then $x, y \in A$, $x \neq y \Rightarrow \exists\, U \in \mathbb{X}$ st wLOG $x \in U$, $y \notin U \Rightarrow x \in U \cap Y$, $y \notin U \cap Y$, but $U \cap Y$ rel. open in A.

If $(\mathbb{X}_\alpha, \tau_\alpha)$ is T_0 $\forall \alpha \in I$. Let $f, g \in \prod_{\alpha \in I} X_\alpha$, $f \neq g$. Then $\exists\, \beta \in I$, st $f(\beta) \neq g(\beta) \Rightarrow \exists\, U_\beta$ st $f(\beta) \in U_\beta$, $g(\beta) \notin U_\beta$

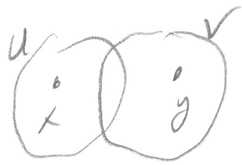
$\Rightarrow \pi_\beta^{-1}(U_\beta)$ is an open set w/ $f \in \pi_\beta^{-1}(U_\beta)$, $g \notin \pi_\beta^{-1}(U_\beta)$.

As a counterexample for quotient spaces, consider $|X| > 1$ with $X_{discrete}$.
Let $x \sim y$ iff $x, y \in X$. Then $X/\sim$ is the trivial topology, and we showed already this is not T_0.

Fact: Any space that is not T_0 can be made into a space that is T_0 by "gluing" the nondistinct points together. That is, $\overline{\{x\}} = \overline{\{y\}}$. Then $X/\sim$ is T_0 if we define $x \sim y$ iff $\overline{\{x\}} = \overline{\{y\}}$.

Pf: Later.

Def^{n}: A space (X, τ) is T_1 iff whenever $x, y \in X$, $x \neq y$, then $\exists\, U, V \in \tau$ st $x \in U$, $y \in V$, $x \notin V$, $y \notin U$.



Clearly, $T_1 \Rightarrow T_0$.

Also, If $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$, Then τ is a topology and (X, τ) is T_0 since $a \neq b$ but $a \in \{a\}$ and $b \notin \{a\}$, $\{a\} \in \tau$, but clearly not T_1.

(1 pt sets are closed)

Equivalencies: (X, τ) is T_1 iff $\forall x \in X$, $\{x\} \in \lambda$
and also iff $\forall A \subseteq X$, $A = \bigcap \{U \in \tau : A \subseteq U\}$.

Pf: If (X, τ) is T_1 and $x \in X$, then for any $y \in \{x\}^c$, $\exists\, U_y \in \tau$ st $y \in U_y$, $x \notin U_y \Rightarrow U_y \cap \{x\} = \emptyset$. Let $U = \bigcup_{y \neq x} U_y$. Then U_y is an open set containing all $y \neq x$ and disjoint from $\{x\}$, so $X - U = \{x\}$, ie $\{x\}$ is closed. (This couldn't be done in a T_0 space since the U_y's guaranteed to exist could contain x and not y; we don't have a say on which is in and which is out)

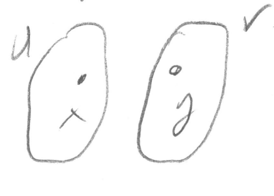
Also, if the 1 pt sets are closed, then for $A \subseteq X$, $A = \bigcap_{x \in A^c} X - \{x\}$, and $X - \{x\}$ is open $\forall x \in X$. [This isn't all the open sets tho?]

If U is an open set st $A \subseteq U$, then $U = X - \bigcup_{\alpha \in I} \{x_\alpha\}$ where $\{x_\alpha\}$ is the collection of singletons which aren't in A and aren't in U.

But $X - \bigcup \{x_\alpha\} = \bigcap \{x_\alpha\}^c = \bigcap_{x \in A^c \cap U^c} X - \{x\}$, so the intersection of open sets containing A is just an intersection of sets of the form $\bigcap_{x \in F \subseteq A^c} X - \{x\}$,

So $\bigcap_{x \in A^c} X - \{x\} = \bigcap \{U \in \tau : A \subseteq U\}$. Finally, if we have that all $A \subseteq X$ are the intersections of open sets containing A, then $\forall x \in X$, $\{x\} = \bigcap \{U \in \tau : x \in U\}$. Thus if $y \neq x$, $y \notin \{x\}$, so $\exists U \in \tau$ st $y \notin U$, otherwise our intersection would have included it. ∎

Def^n: A space (X, τ) is T_2 (or Hausdorff) if $\forall x \neq y \in X$, $\exists U, V \in \tau$ st $x \in U$, $y \in V$, $U \cap V = \emptyset$



Clearly, $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Metric spaces are Hausdorff since if $x \neq y \in (X, \tau)$ let $\varepsilon < \rho(x,y)$. Then $x \in B_\rho(x, \frac{\varepsilon}{2})$, $y \in B_\rho(y, \frac{\varepsilon}{2})$ and if $z \in B_\rho(y, \varepsilon)$, then $\varepsilon < \rho(x,y) \le \rho(x,z) + \rho(y,z) < \rho(x,z) + \frac{\varepsilon}{2} \Rightarrow \rho(x,z) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$
$\Rightarrow z \notin B_\rho(x, \frac{\varepsilon}{2}) \Rightarrow B_\rho(y, \frac{\varepsilon}{2}) \subseteq (B_\rho(x, \frac{\varepsilon}{2}))^c \Rightarrow B_\rho(x, \frac{\varepsilon}{2}) \cap B_\rho(y, \frac{\varepsilon}{2}) = \emptyset$

Define: The <u>cofinite topology</u> on $\mathbb{X}$ by $U \in \tau$ iff U is cofinite; ie $|U^c| < \omega$, or $\mathbb{X} = \emptyset$. $(\mathbb{X}, \tau)$ is a topology: $\emptyset \in \tau$ by construction, $|\mathbb{X} - \mathbb{X}| = |\emptyset| = 0$ so $\mathbb{X} \in \tau$, and if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$,

Then $|(\bigcup U_\alpha)^c| = |\bigcap U_\alpha^c| \leq |U_1^c|$ (since $A \subseteq B \Rightarrow |A| \leq |B|$)

$< \omega$ since $U_1 \in \tau$.

Thus $\bigcup U_\alpha \in \tau$, and if $U, V \in \tau$, then $|(U \cap V)^c| = |U^c \cup V^c|$
$\leq |U^c| + |V^c| = n + m < \omega$ where $n = |U^c|$, $m = |V^c|$, so $U \cap V \in \tau$.

Further, $\boxed{\text{the cofinite top. on any infinite set } \mathbb{X} \text{ is } T_1 \text{ but not } T_2:}$

1 pt sets are closed since $|\mathbb{X}| \geq \omega \Rightarrow |\mathbb{X} - \{x\}| \geq \omega$ for any $x \in \mathbb{X}$,
so $|\{x\}| = 1 \Rightarrow |\mathbb{X} - \{x\}| \geq \omega \Rightarrow \{x\}$ is closed, so $\mathbb{X}$ is T_1.

However, if $U, V \in \tau$, then $U \cap V \neq \emptyset$ since if $U \cap V = \emptyset$,
then $U \subseteq V^c \Rightarrow |U| \leq |V^c| < \omega$, but by $U \in \tau$, $|U^c| < \omega$
$\Rightarrow |U| \geq \omega$, $\#$. So $\mathbb{X}$ cannot be T_2; if $U, V \in \tau$ w/ $x \in U$,
$y \in V$, then $U \cap V \neq \emptyset$.

[I forgot subspace/product space/quotient space facts about T_1]

Thm: Subspaces/products of T_1 spaces are T_1. quotients of T_1 spaces need not be T_1.

Thm: Subspaces/products of T_2 spaces are T_2, quotients need not be.

Pf: T_1-spaces: Let $A \subseteq X$. Then $x, y \in A \Rightarrow x, y \in X \Rightarrow \exists U, V \in \tau$ st $x \in U$, $y \notin U$, $x \in V$, $y \notin V$. $\Rightarrow x \in U \cap A$, $y \notin U \cap A$, $y \in V \cap A$, $x \notin V \cap A$, $U \cap A$, $V \cap A$ rel. open in A, so A is T_1.

Let (X_α, τ_α) be a collection of T_1 spaces. Then $f, g \in \prod X_\alpha$, $f \neq g$ $\Rightarrow \exists_\beta \in I$ st $f(\beta) \neq g(\beta)$ in $X_\beta \Rightarrow \exists U_\beta, V_\beta \in \tau_\beta$ st $f(\beta) \in U_\beta$, $g(\beta) \notin U_\beta \Rightarrow f \in \pi_\beta^{-1}(U_\beta) \wedge g \notin \pi_\beta^{-1}(U_\beta)$, and $g(\beta) \in V_\beta$, $f(\beta) \notin V_\beta \Rightarrow g \in \pi_\beta^{-1}(V_\beta) \wedge f \in \pi_\beta^{-1}(V_\beta)$, so $\prod X_\alpha$ is T_1. The exact same counterexample for T_0 quotients not being T_0 still works, because X_{discrete} is clearly also $T_2 \Rightarrow T_1$, but the quotient $X/\sim$ is not $T_0 \Rightarrow$ not $T_1 \Rightarrow$ not T_2.

T_2-spaces: The exact same argument for subspaces works with the added condition that $U \cap V = \emptyset \Rightarrow (U \cap A) \cap (V \cap A) = \emptyset$. The exact same argument works for products as well, with the added condition that $U_\beta \cap V_\beta = \emptyset \Rightarrow \pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \emptyset$. $\boxtimes$

Equivalencies for T_2: TFAE: (1) X is T_2

☆ (2) limits are unique ☆

(that is, if $\{x_\lambda\}_{\lambda \in \Lambda}$ is a net w/ $x_\lambda \to x$ and $x_\lambda \to y$ then $x = y$, and if $\mathcal{F}$ is a filter on X st $\mathcal{F} \to x$ and $\mathcal{F} \to y$ then $x = y$)

(3) The diagonal $\{(x,x) : x \in X\}$ is closed in $X \times X$.

Pf: S. X is T_2 and $\mathcal{F}$ a filter w/ $\mathcal{F} \to x$ and $\mathcal{F} \to y$. S $x \neq y$. Then $\exists U, V \in \tau$ st $x \in U$, $y \in V$, $U \cap V = \emptyset$. Since U is a nbhd of x, $U \in \mathcal{F}$ since $\mathcal{F} \to x$ and $V \in \mathcal{F}$ since $\mathcal{F} \to y$, and $\mathcal{F}$ is a filter so $U \cap V \in \mathcal{F}$ but $U \cap V = \emptyset$, so $\emptyset \in \mathcal{F}$, ※. So $x = y$.

Before giving the pf for nets, we note an equivalence between nets and filters.

Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net. Then the filter generated by the filter base $\mathcal{B} = \{B_{\lambda_0} : \lambda_0 \in \Lambda\}$ where $B_{\lambda_0} = \{x_\lambda : \lambda \geq \lambda_0\}$ is called the <u>filter generated by $\{x_\lambda\}$</u>. This is a filter base: If $B_{\lambda_0}, B_{\lambda_1} \in \mathcal{B}$,

Then by directedness $\exists \lambda_2 \geq \lambda_1 \wedge \geq \lambda_0 \in \Lambda$, so

$B_{\lambda_2} = \{x_\lambda : \lambda \geq \lambda_2\} \subseteq \{x_\lambda : \lambda \geq \lambda_1\}$ and $\subseteq \{x_\lambda : \lambda \geq \lambda_0\}$

Since by transitivity if $\lambda \geq \lambda_2$ and $\lambda_2 \geq \lambda_{1/0}$ then $\lambda \geq \lambda_{1/0}$.

So $B_{\lambda_2} \subseteq B_{\lambda_0} \cap B_{\lambda_1}$.

Also, if $\mathcal{F}$ is a filter on X, then let $\Lambda_{\mathcal{F}} = \{(x, F) : x \in F \in \mathcal{F}\}$

Then define the directed partial ordering $(x_1, F_1) \leq (x_2, F_2)$ iff $F_2 \subseteq F_1$

(we showed this was directed in an earlier proof). Then the map $p : \Lambda_{\mathcal{F}} \to X$ defined by $p((x, F)) = x$ is a net in X. Call it the <u>net based on $\mathcal{F}$</u>.

<u>Thm</u> (a) A filter $\mathcal{F} \to x \in X$ iff the net based on $\mathcal{F}$ converges to x.

(b) A net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to x iff the filter generated by $\{x_\lambda\}$ converges to x

<u>Pf</u> (a) $\mathcal{F} \to x$. Let U be a nbhd of x. Then $U \in \mathcal{F}$. Let $p \in U$. Then $(p, U) \in \Lambda_{\mathcal{F}}$, and if $(q, F) \geq (p, U)$ then $q \in F \subseteq U$, ie $\Lambda_{\mathcal{F}}$ is eventually in U, ie it converges to x. Conversely, if the net based on $\mathcal{F}$ converges to x, then if U is a nbhd of x, then $\exists (p_0, F_0) \in \Lambda_{\mathcal{F}}$ st if $(p, F) \geq (p_0, F_0)$ then $p \in U$. But then $F_0 \subseteq U$: If not, then $F_0 \cap U^c \neq \emptyset \Rightarrow \exists q \in F_0 - U \Rightarrow (q, F_0) \geq (p_0, F_0)$, but $q \notin U$, $\times$. Thus $U \in \mathcal{F}$, ie $\mathcal{F} \to x$.

(5) The net $\{X_\lambda\}_{\lambda\in\Delta} \to x$ iff each nbhd of x contains a tail of X_λ. These tails are the base for the filter generated by $\{X_\lambda\}$, so our result follows. ☒

getting back to T_2-stuff, since filters converge to unique limits so do nets, since if a net converges to 2 distinct pts, then the filter generated by it converges to two distinct pts, which cannot happen. Thus, we have $(1)\Rightarrow(2)$. For $(2)\Rightarrow(3)$, $\mathcal{S}$ bwoc that Δ is not closed. Then Δ is a proper subset of $\overline{\Delta}$, ie $\exists (x,y)\in \mathbb{X}\times\mathbb{X}$ st $(x,y)\in\overline{\Delta}$ but $(x,y)\notin\Delta$. $(x,y)\in\overline{\Delta} \Rightarrow \exists$ a net $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda} \subseteq \Delta$ st $(x_\lambda,y_\lambda)\to(x,y)$ with $(x,y)\notin\Delta \Rightarrow x\neq y$. But since the net is in Δ, $x_\lambda = y_\lambda\ \forall\lambda$, ie the component nets $\{x_\lambda\}_{\lambda\in\Delta}$ and $\{y_\lambda\}_{\lambda\in\Delta}$ are really the same. thus $\{x_\lambda\}_{\lambda\in\Delta}\to x$ and $\{x_\lambda\}_{\lambda\in\Delta} = \{y_\lambda\}_{\lambda\in\Delta}\to y$ so our net converges to two distinct points in $\mathbb{X}$, contradiction. So $\Delta=\overline{\Delta}$ ie Δ is closed. Finally, for $(3)\Rightarrow(1)$, $\mathcal{S}$ Δ is closed. If $x\neq y$ in $\mathbb{X}$, then $(x,y)\notin\Delta$, and hence $\exists$ a basic nbhd $U\times V$ of (x,y) in $\mathbb{X}\times\mathbb{X}$ st $(U\times V)\cap\Delta=\emptyset$. But this means that $\forall x\in U$, $\forall y\in V$, $x\neq y$, since otherwise $x=y \Rightarrow (x,y)\in\Delta \wedge (x,y)\in(U\times V)$. So $U\cap V=\emptyset$, ie $\mathbb{X}$ is Hausdorff. ☒

Fact: If $(\mathbb{X},\tau)$ is T_2 and $A\subseteq\mathbb{X}$, then $A'\in\lambda$.

Pf: HW problem

Thm: For any linear order, $(\mathbb{X},\leq)$ with the order topology is always T_2.

Pf:

Let $x, y \in X$, $x \neq y$. WLOG $x < y$. If $\exists z$ st $x < z < y$, then let $U = (-\infty, z)$, $V = (z, \infty)$. Then clearly $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Else, let $U = (-\infty, y)$, $V = [y, \infty) = (x, \infty) \leftarrow$ so open.

Then $x \in U$, $y \in V$, $U \cap V = \emptyset$. $\boxtimes$

Thus $[0, \omega_1)$ and $[0, \omega_1]$ are T_2.

Thm (AC): Let $X = \prod_{\alpha \in I} X_\alpha$ be an uncountable product of spaces. Then if $|X_\alpha| \geq 2$ for each α, and (X_α, τ_α) is T_1 $\forall \alpha$, then X is not 1^{st} ctble. [and thus not a metric space and not 2^{nd} ctble]

Pf: Let $x \in X$, & $\mathcal{B}_x = \{B_n\}_{n \in \omega}$ is a ctble base at x. WLOG may assume each B_n is a basic open set in the typical product base. [If $B_n \in \mathcal{B}_x$ then since $x \in X$, $\exists \tilde{B}_n$ basic open st $x \in \tilde{B}_n \subseteq B_n$, then $\tilde{\mathcal{B}}_x = \{\tilde{B}_n\}_{n \in \omega}$ is a nbhd base]. Since each B_n is basic open, $\exists F_n \subseteq I$ st $B_n = \bigcap_{\beta \in F_n} \pi_\beta^{-1}(U_{n,\beta})$. $\bigcup_{n \in \mathbb{N}} F_n$ is a ctble union of finite sets, so it's ctble. Let $\alpha_0 \in I - \bigcup_{n \in \omega} F_n$. (nonempty since I is unctble).

Since $|X_{\alpha_0}| \geq 2$, X_{α_0} is T_1, $\exists V \in \tau_{\alpha_0}$ st $x(\alpha_0) \in V$ and $V \neq X_{\alpha_0}$ (since $\exists y \in X_{\alpha_0}$ st $y \neq x(\alpha_0)$, $y \notin V$. Consider the basic open set $\pi_{\alpha_0}^{-1}(V)$. Clearly $x \in \pi_{\alpha_0}^{-1}(V)$ since $x(\alpha_0) \in V$. However, we claim that $\neg\exists B_n \in \mathcal{B}_x$ st $x \in B_n \subseteq \pi_{\alpha_0}^{-1}(V)$: $\alpha_0 \notin F_n$ by design, and so define $y \in X$ by

$$y(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha \in F_n \\ z \notin V & \text{if } \alpha = \alpha_0 \\ \text{anything else} & \text{otherwise.} \end{cases}$$

Then $y \in B_n$ but $y \notin \pi_{\alpha_0}^{-1}(V)$, so $B_n \not\subseteq \pi_{\alpha_0}^{-1}(V)$, so $\mathcal{B}_x$ isn't a nbhd base, ※. Thus $\prod_{\alpha \in I} X_\alpha$ is not 1^{st} ctble. $\boxtimes$

So as a consequence, since $[0,1]_{std} \subseteq \mathbb{R}_{std}$ which is a metric space and thus $T_2 \Rightarrow T_1$, $[0,1]_{std}$ is T_1. Thus $\prod_{\alpha \in I} [0,1]_{std}$ is 1^{st} ctble iff I is ctble.

<u>Def$^{\underline{n}}$</u>: A space (X, τ) is <u>regular</u> iff whenever $A \in \mathfrak{C}$ and $x \notin A$, $\exists$ disjoint $U, V \in \tau$ st $x \in U$ and $A \subseteq V$. (ie we can separate points from closed sets, ie closed sets are topologically distinguishable).

A space is T_3 if it is regular and T_1 (we need T_1 since regular spaces don't have to be even T_0: The trivial topology is an example of a regular space which is not T_0)

A T_3 space is T_2, since if $x, y \in X$, $\{x\}$ and $\{y\}$ are closed, So $\exists\, U, V \in \tau$ st $x \in U$, $\{y\} \subseteq V \Rightarrow y \in V$, and $U \cap V = \emptyset$.

[Also, the definition of T_3 could have equivalently been regular + T_2 or regular + T_0: regular + $T_2 \Rightarrow T_3$ is obv, and regular and $T_0 \Rightarrow$ if $x \neq y \in X$, $\exists\, U \in \tau$ st WLOG $x \in U$, $y \notin U$. Then $\{y\} \in \mathfrak{C}$, so $\exists$ open V, W st $x \in V$, $\{y\} \subseteq W \Rightarrow y \in W$, and $V \cap W = \emptyset$. $x \in U$ and $x \in V \Rightarrow x \in U \cap V \in \tau$, and $(U \cap V) \cap (W) \subseteq V \cap W = \emptyset$ $\Rightarrow U \cap V$ and W are disjoint, so regular + $T_2 \Leftrightarrow$ regular + $T_1 \Leftrightarrow$ regular + T_0]

What's a T_2 space which is not T_3?

Let $(\mathbb{R}, \tau)$ be the topology defined in the following way:

If $x \in \mathbb{R}$, $x \neq 0$, define the nbhd system $\mathcal{U}_x$ = the standard nbhd system in $\mathbb{R}_{std}$.

and if $x = 0$, define $\mathcal{U}_0 = \{U - A : U \text{ contains a nbhd of } 0 \text{ in } \tau_{std}\}$

where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

The topology $\tilde{\tau}$ is defined by $U \in \tilde{\tau}$ iff $\forall x \in U$, $\exists V \in \mathcal{U}_x$ st $V \subseteq U$. Defining a nbhd system at each pt is sufficient for defining a topology. Clearly $\tau_{std} \subseteq \tilde{\tau}$, since all standard nbhds of 0 are still nbhds in $\tilde{\tau}$ $(0 \in U \in \tau_{std} \Rightarrow 0 \in U - A \subseteq U \Rightarrow U \in \tilde{\tau})$ So since τ_{std} is T_2, so is $\tilde{\tau}$. We claim $\tilde{\tau}$ is not T_3 though: $A \in \lambda$, since $A^c = (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup \dots \in \tilde{\tau}$. Further, $0 \notin A$. But if U is any open set containing A and V is an open set containing 0, then $U \cap V \neq \emptyset$. To show this, first note that if U is a standard nbhd of 0, then already $U \cap V \neq \emptyset$ since $U \cap A \neq \emptyset$, since U contains some $(-\varepsilon, \varepsilon)$ and if $\frac{1}{n} < \varepsilon$, then $\frac{1}{n} \in A$ and $\frac{1}{n} \in (-\varepsilon, \varepsilon)$, so $U \cap A \neq \emptyset$. Otherwise, U has to contain some $(-\varepsilon, \varepsilon) - A$. However, since $V \supseteq A$, pick again $n \in \mathbb{N}$ st $\frac{1}{n} < \varepsilon$. Then V is open and $\frac{1}{n} \in V \Rightarrow \exists (\frac{1}{n} - \varepsilon_0, \frac{1}{n} + \varepsilon_0) \subseteq V$. Let $y \in (\frac{1}{n} - \varepsilon_0, \varepsilon)$ st $y \neq \frac{1}{m}$ for any $m \in \mathbb{N}$. Then $y \notin A$ and $y \in (\frac{1}{n} - \varepsilon_0, \varepsilon) \subseteq (-\varepsilon, \varepsilon)$ $\Rightarrow y \in U$, and $y \in (\frac{1}{n} - \varepsilon_0, \frac{1}{n} + \varepsilon_0) \subseteq V$ so $y \in V$ and thus $U \cap V \neq \emptyset$. So τ is not T_3. $\boxed{//}$

Thm (Equivalencies to being Regular): TFAE

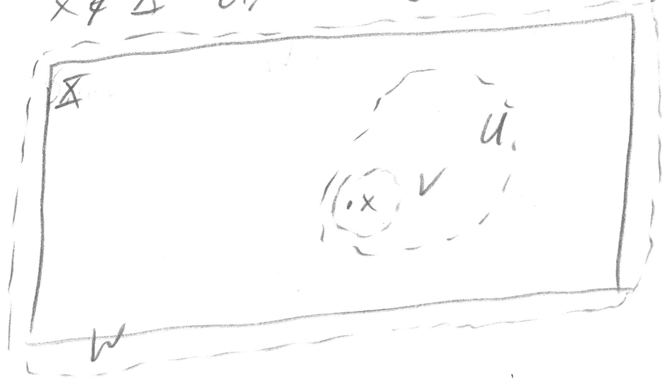
(a) (X, τ) is Regular

(b) If $U \in \tau$ and $x \in U$, then $\exists V \in \tau$ st $x \in V$ and $\bar{V} \subseteq U$ (V is a *shrinking* of U)

(c) for each $x \in X$, $\exists$ a filter base of closed sets which generates the nbhd system $\mathcal{U}_x$. [where $\forall F \in$ the filter base for a pt x, $x \in F^o$]

[Note that calling V a shrinking of $\mathcal{U}$ is justified since if $V = \mathcal{U}$ then $\overline{V} \subseteq \mathcal{U} \iff \overline{V} \subseteq V$ and $V \subseteq \overline{V} \Rightarrow V = \overline{V} \Rightarrow V$ is closed. So as long as $\mathcal{U}$ isn't clopen, V is properly contained in $\mathcal{U}$.]

Pf: (a) $\Rightarrow$ (b). If X is regular, $\mathcal{U} \in \tau$, $x \in \mathcal{U}$. Then $X - \mathcal{U}$ is closed, $x \notin X - \mathcal{U}$, so $\exists V, W \in \tau$ st $x \in V$, $X - \mathcal{U} \subseteq W$, $V \cap W = \emptyset$.



$X - \mathcal{U} \subseteq W \iff X - W \subseteq \mathcal{U}$

Further, $V \cap W = \emptyset \iff V \subseteq X - W$

Thus $\overline{V} \subseteq X - W \subseteq \mathcal{U}$

So $x \in V$ and $\overline{V} \subseteq \mathcal{U}$. ☒

(b) $\Rightarrow$ (c): Let Ξ_x be a nbhd base at $x \in X$. Then for each $\mathcal{U} \in \Xi_x$, $\exists V \in \tau$ st $x \in \overline{V} \subseteq \mathcal{U}$. The collection of all such $\overline{V}$ is clearly a filter base for the nbhd system since if $A \subseteq X$ st $x \in A^\circ$, then $\exists \mathcal{U} \in \Xi_x$ st $x \in \mathcal{U} \subseteq A \Rightarrow x \in \overline{V} \subseteq \mathcal{U} \subseteq A$.

(c) $\Rightarrow$ (a): Let $A \in \lambda$, with $x \notin A$. Then $X - A$ is an (open) nbhd of x, so $\exists B \in \lambda$ st $x \in B^\circ \subseteq B \subseteq X - A$. Also, $A \subseteq X - B$, and $B \in \lambda \Rightarrow X - B \in \tau$, so B° and $X - B$ are open sets w/ $x \in B^\circ$, $A \subseteq X - B$, and $B^\circ \cap X - B \subseteq B \cap (X - B) = \emptyset$. ☒

Thm:
- Subspaces/products of Regular/T_3 spaces are Regular/T_3.
- Quotients of T_3 spaces need not be regular

Pf: If X is regular, $Y \subseteq X$, and A rel. closed in Y, then $A = B \cap Y$, where B is closed in X. If $y \in Y$, $y \notin A$, then $y \notin B$, so $\exists$ open $\mathcal{U}, V$ in X st $y \in \mathcal{U}$, $B \subseteq V$, $\mathcal{U} \cap V = \emptyset$. But then $y \in \mathcal{U} \cap Y$, $A = B \cap Y \subseteq V \cap Y$, and $(\mathcal{U} \cap Y) \cap (V \cap Y) \subseteq \mathcal{U} \cap V = \emptyset$, so Y is regular as a subspace of X.

T_2 stuff carries down from earlier. If $(\mathbb{X}_\alpha, \tau_\alpha)$ is a collection of regular/nonempty spaces, let $f \in \prod \mathbb{X}_\alpha$. Consider a basic nbhd of $f \in \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$. U_{α_i} is a nbhd of $f(\alpha_i)$ for each $i = 1, \dots, n$, so since $\mathbb{X}_\alpha$ is regular, $\exists$ a shrinking V_{α_i} st $x \in V_{\alpha_i} \subseteq \overline{V}_{\alpha_i} \subseteq U_{\alpha_i}$. Then $\pi_{\alpha_i}^{-1}(\overline{V}_{\alpha_i}) \cap \dots \cap \pi_{\alpha_n}^{-1}(\overline{V}_{\alpha_n})$ is a closed nbhd of f [If $F_\alpha = \mathbb{X}_\alpha - U_\alpha$ for some open U_α, then $\prod \mathbb{X}_\alpha - \pi_\alpha^{-1}(U_\alpha) = \{f : f(\alpha) \in F_\alpha\} = \pi_\alpha^{-1}(F_\alpha)$ so $\pi_\alpha^{-1}(F_\alpha)$ is closed in the product if F_α is closed in $\mathbb{X}_\alpha$].

Thus our open nbhd of f has a shrinking, so $\prod \mathbb{X}_\alpha$ is regular.

__Def^{n}__: A space is __completely regular__ if $\forall A \in \mathcal{C}$ and $x \notin A$, there is a cts function $f : \mathbb{X} \to [0,1]_{std}$ st $f(x) = 0$ and $f(A) = 1$.

We say that f __separates__ A and x. A space which is completely regular and T_1 is a $T_{3\frac{1}{2}}$ space (also called __Tychonoff__).

Completely Regular $\Rightarrow$ Regular, since if A closed, $x \notin A$, completely regular $\Rightarrow \exists f : \mathbb{X} \to [0,1]$ st $f(x) = 0$, $f(A) = 1 \Rightarrow f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are open in $\mathbb{X}$ since $(\frac{1}{2}, 1]$ and $[0, \frac{1}{2})$ are rel. open in $[0,1]$, and clearly these are disjoint, with $A \subseteq f^{-1}((\frac{1}{2}, 1])$ and $x \in f^{-1}([0, \frac{1}{2}))$. Thus $T_{3\frac{1}{2}} \Rightarrow T_3$. However, $T_3 \not\Rightarrow T_{3\frac{1}{2}}$, but there are __very__ few examples. So $T_3 \overset{\text{almost}}{\Longrightarrow} T_{3\frac{1}{2}}$.

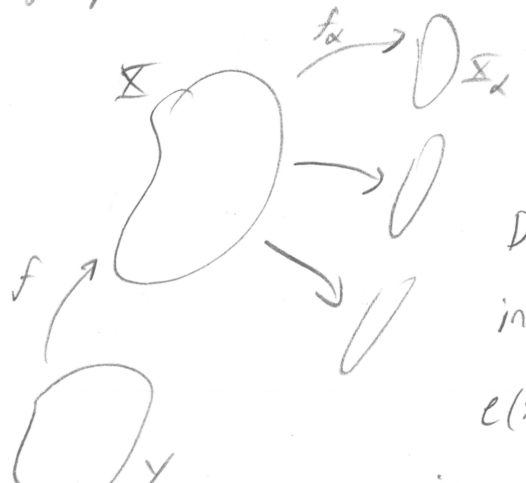
The ones that exist are contrived and not worth the effort.

Thm: Every metric space is $T_{3\frac{1}{2}}$

Pf: Let $A \subseteq X$, $x \notin A$. Then $f(y) = \rho(y, A)$ is a well defined cts function (nts) from X to $\mathbb{R}$ st $f(A) = 0$ and $f(x) \neq 0$. This is enough since $\frac{\rho(x)}{1+\rho(x)}$ is a metric bounded between 0 and 1, so it can be assumed $f: X \to [0,1]$ WLOG, and if we further redefine ρ by letting $\tilde{\rho}(x) = \frac{\rho(x)}{\rho(x, A)}$ then $\tilde{\rho}(x) = 1$, and it's still a metric. $\boxed{\not\!/}$

Journey to Cubeness

Recall that if (X_α, τ_α) is a collection of spaces and X has the weak topology induced by a collection $\{f_\alpha\}_{\alpha \in A}$, then $f: Y \to X$ is cts iff $f_\alpha \circ f$ is cts $\forall \alpha \in I$.



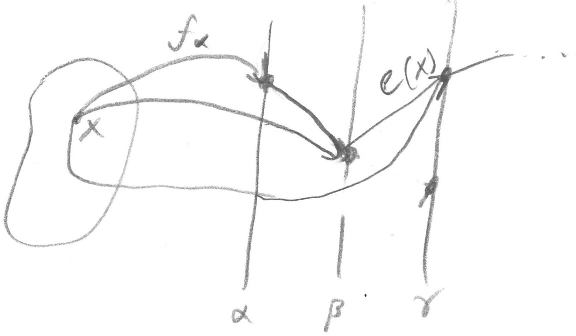
We turn to the question: When can X be embedded into $\prod_{\alpha \in I} X_\alpha$?

Define the <u>evaluation map</u> $e: X \to \prod X_\alpha$ induced by the collection $\{f_\alpha\}$ by

$$e(x) = f \in \prod X_\alpha \quad \text{where } f(\alpha) = f_\alpha(x)$$

ie $\quad [e(x)](\alpha) = f_\alpha(x)$

To clarify, each pt in X maps to a pt $f_\alpha(x)$ in each X_α



this defines a function $[e(x)] \in \prod X_\alpha$ where $[e(x)](\alpha) = f_\alpha(x)$ ie e is a function which sends pts $x \in X$ to their corresponding functions in $\prod X_\alpha$.

Note that in the product $\prod X_\alpha$, the $\pi_\beta : \prod X_\alpha \to X_\beta$ are cts open maps, and that $f: Y \to \prod X_\alpha$ is cts iff $\pi_\alpha \circ f : Y \to X_\alpha$ is cts for each α. Therefore, $e: X \to \prod X_\alpha$ is cts iff $\pi_\alpha \circ e$ is cts for each $\alpha \in I$.

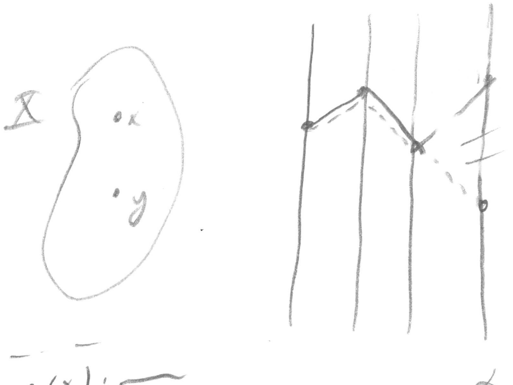
But $(\pi_\alpha \circ e)(x) = \pi_\alpha([e(x)]) = [e(x)](\alpha) = f_\alpha(x)$.

So if X is inheriting the weak topology from the $\{f_\alpha\}$, then the f_α are all cts, and thus $\pi_\alpha \circ e$ is cts $\forall \alpha \in I$, which means e is cts.

Recall $\varphi : X \to Y$ is an embedding if it is 1-1 and a homeomorphism from X to $\varphi(X)$, ie it is a continuous open map. $e: X \to \prod X_\alpha$ is now known to be cts as long as X has the weak topology from $f_\alpha : X \to X_\alpha$.

Thus showing e is an embedding amounts to showing it is 1-1 and an open map.

We say the collection $\{f_\alpha\}_{\alpha \in I}$ <u>seperates points</u> if $\forall x \neq y \in X$, $\exists \alpha \in I$ st $f_\alpha(x) \neq f_\alpha(y)$



$e(x)$: —

$e(y)$: - - -

Thm: Let (X, τ) be a space, $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$ be a collection of spaces, and $\{f_\alpha : X \to X_\alpha\}_{\alpha \in I}$ a collection of functions.

Then

X embeds into $\prod_{\alpha \in I} X_\alpha$ iff

(1) τ is the weak topology induced by the $\{f_\alpha\}$

and (2) The collection $\{f_\alpha\}$ seperates points.

Pf: We claim the evaluation map $e : X \to \prod X_\alpha$ is our embedding (we're doing $\Leftarrow$ directly). We already have that e is cts.

To show it is 1-1, let $x, y \in X$, $x \neq y$. Then since the collection $\{f_\alpha\}$ seperates pts, $\exists \alpha \in I$ st $f_\alpha(x) \neq f_\alpha(y)$.

Hence $[e(x)](\alpha) \neq [e(y)](\alpha) \Leftrightarrow e(x) \neq e(y)$. Hence e is 1-1.

To show it is open, let $U \in \tau$. Wts $V = e(U)$ is open.

Let $e(x) \in V$, ie $x \in U$. Since X has the weak top. induced by the $\{f_\alpha\}$, a subbasis for X is $\mathcal{S} = \{ f_\alpha^{-1}(U) : U \in \tau_\alpha, \alpha \in I \}$.

Thus, since $U \in \tau$, $\exists \alpha_1, \dots, \alpha_n$, and $U_{\alpha_1}, \dots, U_{\alpha_n}$ st $x \in f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap f_{\alpha_n}^{-1}(U_{\alpha_n}) \subseteq U$. Let $B_x = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$

B_x is a basic open set in $\prod X_\alpha$. Note that for each $i = 1, \dots, n$,

$\pi_{\alpha_i}(e(x)) = (\pi_{\alpha_i} \circ e)(x) = f_{\alpha_i}(x) \in U_{\alpha_i}$, so $e(x) \in B_x$. Thus, $\forall x \in U$, $\exists$ a basic open B_x in $\prod X_\alpha$ st $x \in B_x$. Nts $B_x \subseteq V$. if $e(y) \in B_x$, then $e(y) \in \pi_\alpha^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_\alpha^{-1}(U_{\alpha_n})$, ie $\pi_{\alpha_i}^{-1}(e(y)) = f_{\alpha_i}(y) \in U_{\alpha_i}$ for each $i = 1, \dots, n$, ie $y \in U$. Thus $e(y) \in V$, ie $B_x \subseteq V$. so V is open in the product. $\therefore$ e is an embedding.

Conversely, suppose $e : X \to \prod X_\alpha$ is an embedding. The chain of implications in showing $\{f_\alpha\}$ separates pts $\Rightarrow e$ is 1-1 was in fact $\Leftrightarrow$, so the collection $\{f_\alpha\}_{\alpha \in I}$ seperates pts.

To show X has the weak topology, let $U \in \tau$. Then $e : X \to e(X)$ is a homeomorphism so $e(U)$ is open in $e(X)$.

Thus $\forall x \in e(U)$, $\exists$ a basic open set st $e(x) \in \left(\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})\right) \cap e(X)$

$\subseteq e(U)$,

Let $B = f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n})$. This is basic open in the weak topology. Then if $y \in B$, then $f_{\alpha_i}(y) \in U_{\alpha_i}$ for each i.

But $f_{\alpha_i}(y) = (\pi_{\alpha_i} \circ e)(y)$, so $e(y) \in \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$

Thus $B \subseteq U$. $\boxed{\checkmark}$

<u>Def</u>: A collection $\{f_\alpha : X \to X_\alpha\}$ is said to <u>separate pts from closed sets</u> iff whenever F is closed in X, $x \notin F$, $\exists \alpha \in I$ st $f_\alpha(x) \notin \overline{f_\alpha(F)}$

<u>Thm</u>: A collection $\{f_\alpha : X \to X_\alpha\}$ of continuous functions separates pts from closed sets iff the collection $\{f_\alpha^{-1}(U_\alpha) : \alpha \in I, U_\alpha \in \tau_\alpha\}$ is a base for the topology on X.

<u>Pf</u>: $\Rightarrow$ If $\{f_\alpha\}_{\alpha \in I}$ separates pts from closed sets, let $U \in \tau$, $x \in U$.

Let $F = X - U$. F is closed, and $x \notin F$, so $\exists \alpha \in I$ st $f_\alpha(x) \notin \overline{f_\alpha(F)}$

Thus $\exists W \in \tau_\alpha$ st $f_\alpha(x) \in W$ and $W \cap \overline{f_\alpha(F)} = \emptyset$.

Clearly, $x \in f_\alpha^{-1}(W)$. Since f_α is cts, $f_\alpha(F) = f_\alpha(F) \subseteq \overline{f_\alpha(F)}$

Thus $X_\alpha - \overline{f_\alpha(F)} \subseteq X_\alpha - f_\alpha(F)$. $W \cap \overline{f_\alpha(F)} = \emptyset$ means

$W \subseteq X_\alpha - \overline{f_\alpha(F)} \subseteq X_\alpha - f_\alpha(F)$. Then $f_\alpha^{-1}(W) \subseteq f_\alpha^{-1}(X_\alpha - f_\alpha(F))$

$= f_\alpha^{-1}(X_\alpha) - f_\alpha^{-1}(f_\alpha(F)) = X - f_\alpha^{-1}(f_\alpha(F)) \subseteq X - F$

(since $F \subseteq f^{-1}(f(F))$) $\therefore$ $f_\alpha^{-1}(W) \subseteq X - F = U$. So the collection

$\{f_\alpha^{-1}(W) : \alpha \in I, W \in \tau_\alpha\}$ is a base for (X, τ).

$\Leftarrow$ Now assume that our base in question is in fact a base.

Let F closed in X, $x \notin F$. Let $U = X - F$. Then by hyp.,

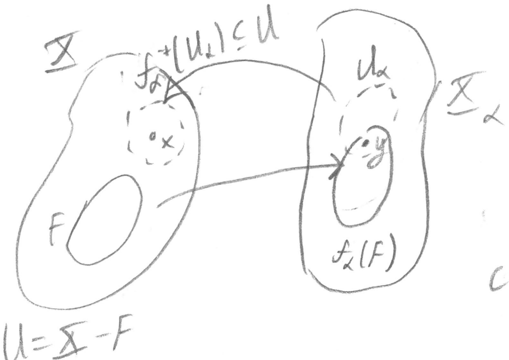
$\exists \alpha \in I$, $U_\alpha \in \tau_\alpha$ st $x \in f_\alpha^{-1}(U_\alpha) \subseteq U$. Clearly $f_\alpha(x) \in U_\alpha$.

If $U_\alpha \cap f_\alpha(F) = \emptyset$, then we'll have that $f_\alpha(x) \notin \overline{f_\alpha(F)}$.

(since this is the negation of $\forall U \in \tau$ w/ $x \in U$, $U \cap F \neq \emptyset$)

& $y \in U_\alpha \cap f_\alpha(F)$. Then $\exists z \in F$ st $f_\alpha(z) = y \in U_\alpha$.

$z \in F \Rightarrow z \notin U \Rightarrow z \notin f_\alpha^{-1}(U_\alpha) \Rightarrow f_\alpha(z) \notin U_\alpha$, $\divideontimes$.



So $U_\alpha \cap f_\alpha(F) = \emptyset$, ie $f_\alpha(x) \notin \overline{f_\alpha(F)}$, and so we have that the $\{f_\alpha\}$ family separates pts from closed sets. $\boxtimes$

<u>Corollary</u>: If $\{f_\alpha : \mathbb{X} \to \mathbb{X}_\alpha\}$ is a collection of cts functions which separate pts from closed sets, then the topology on $\mathbb{X}$ is the weak topology induced by the collection $\{f_\alpha\}$.

<u>Pf</u>: Note that if $\mathcal{B}$ is a base for τ_1 and a subbase for τ_2, then $\tau_1 \subseteq \tau_2$, since $\mathcal{B}' = \{\bigcap_{i=1} B_i : i < \omega, B_i \in \mathcal{B}\}$ is a base for τ_2 and the collection $\mathcal{B} \subseteq \mathcal{B}' \Rightarrow \tau_1 \subseteq \tau_2$. We already know that $\{f_\alpha^{-1}(U_\alpha) : \alpha \in I, U_\alpha \in \tau_\alpha\}$ is a subbase for the weak topology on $\mathbb{X}$ induced by the $\{f_\alpha\}$. Since by def^n the weak topology is the coarsest topology st $f_\alpha : \mathbb{X} \to \mathbb{X}_\alpha$ are all cts, if we suppose hyp. that the $f_\alpha : \mathbb{X} \to \mathbb{X}_\alpha$ are all cts, then the topology on $\mathbb{X}$, call it τ, is finer than τ_{weak}, ie $\tau_{weak} \subseteq \tau$. But if the f_α also separate pts from closed sets, then the subbase for τ_{weak} is a base for τ, so $\tau \subseteq \tau_{weak}$, ie $\tau = \tau_{weak}$. $\boxtimes$

Now, suppose the collection $\{f_\alpha : X \to X_\alpha\}$ separates pts from closed sets. If X is T_1, then the singleton sets are closed, so the collection will also separate pts.

Thus, we finally arrive at this thm:

Thm: If X is a T_1 space, $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ a collection of spaces, and $\{f_\alpha : X \to X_\alpha\}$ a collection of cts functions which separate pts from closed sets, then the evaluation map $e : X \to \prod X_\alpha$ is an embedding. ✡

Pf: Our hypothesis gives that the topology on X must already be the weak topology induced by the $\{f_\alpha\}$ by the pvs corollary. If X is T_1, then separating pts from closed sets implies separating points, so these two properties together are necessary and sufficient for e to be an embedding. $\boxed{//}$

We can slightly reformulate our def^{n} of completely regular: If $F \in \mathcal{C}$ and $x \notin F$, from earlier, X is completely regular iff $\exists$ a cts function $f : X \to [0,1]$ st $f(x) = 0$ $f(F) = \{1\}$. Since $\overline{\{1\}} = \{1\}$, clearly $f(x) \notin \overline{f(F)}$.

Let $C^*(X)$ denote the collection of all cts, bounded, real valued functions on X. From what we just wrote, if X is completely regular, then the collection $C^*(X)$ separates points from closed sets. By our previous results, this means that X has the weak topology induced by $C^*(X)$.

Conversely, suppose that X has the weak topology induced by $C^*(X)$. Let U be open in X, with $x \in U$. Then there exist functions $f_1, \dots, f_n$ in $C^*(X)$ and (WLOG) subbasic open sets $V_1, \dots, V_n$ st ...

$\therefore$ $x \in f_1^{-1}(V_1) \cap \dots \cap f_n^{-1}(V_n) \subseteq U$. Note that the collection $\mathcal{S} = \{V : V = (x, \infty) \text{ or } V = (-\infty, x),\ x \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}_{std}$, since any $(a,b) \in \mathcal{B}_{std}$ is just $(-\infty, b) \cap (a, \infty)$. Thus each V_i is of the form (a_i, ∞) or $(-\infty, a_i)$ for some $a_i \in \mathbb{R}$.

if $V_i = (-\infty, a_i)$, then $f_i^{-1}(V_i) = f_i^{-1}((-\infty, a_i)) = (-f_i)^{-1}((-a_i, \infty))$, and if f is cts, bdd, then $-f$ is also cts, bdd, so WLOG we can assume each V_i has the form (a_i, ∞).

For each $i = 1, \dots, n$, let $g_i(x) = \max\{f_i(x) - a_i, 0\}$.

Then clearly g_i is nonnegative, and observe that $g_i^{-1}((0,\infty))$
$= \{x \in X : f_i(x) > a_i\} = f_i^{-1}((a_i, \infty))$. Hence,

$$f_1^{-1}(V_1) \cap \dots \cap f_n^{-1}(V_n) = g_1^{-1}((0,\infty)) \cap \dots \cap g_n^{-1}((0,\infty)).$$

and $x \in g_1^{-1}((0,\infty)) \cap \dots \cap g_n^{-1}((0,\infty)) \subseteq U$.

Let $g(z) := \prod_{i=1}^{n} g_i(z)$ (the actual, like, multiplying of the numbers)

Now, observe that since $x \in f_i^{-1}((a_i, \infty))$ for each i, $f_i(x) > a_i$, so $g_i(x) > 0$. Thus $g(x) > 0$, ie $x \in g^{-1}((0,\infty))$. Moreover, if $g(y) > 0$, then each $g_i(y) > 0$, (since they're nonnegative)

and thus $y \in g_1^{-1}((0,\infty)) \cap \dots \cap g_n^{-1}((0,\infty))$

Hence, $g_1^{-1}((0,\infty)) \cap \dots \cap g_n^{-1}((0,\infty)) \supseteq g^{-1}((0,\infty))$, ie $g^{-1}((0,\infty)) \subseteq U$

ie $g^{-1}((0,\infty)) \cap (X - U) = \emptyset$, but then since g is nonnegative, $\forall y \in X - U$, $g(y) = 0$. So $g(X - U) = 0$, $g(x) \neq 0$.

Any closed set not containing x is the complement of an open set containing x, g is cts since max's and products of cts functions are cts. So X is completely regular!

This proves the following:

Thm: A space X is completely regular iff it has the weak topology induced by its family $C^*(X)$ of real valued cts functions.

Any product of closed, bounded intervals is called a <u>cube</u>.

Since any interval $[a,b] \subseteq \mathbb{R}$ is homeomorphic to $[0,1]$, any cube is homeomorphic to $\prod_{\alpha \in I} [0,1]_\alpha$ for some $I \in ON$.

We showed that every metric space is $T_{3\frac{1}{2}}$. Further, any product of $T_{3\frac{1}{2}}$ spaces is $T_{3\frac{1}{2}}$, so all cubes are $T_{3\frac{1}{2}}$. Further still, subspaces of $T_{3\frac{1}{2}}$ spaces are still $T_{3\frac{1}{2}}$, so all subspaces of cubes are $T_{3\frac{1}{2}}$. Conversely...

$\mathcal{S}$ X is $T_{3\frac{1}{2}}$. Then it's T_1. Further, it's completely regular, so it's collection of real valued, bdd, cts functions $C^*(X)$ separates points from closed sets. Each $f \in C^*(X)$ maps onto some closed, bdd interval $[a,b] = I_f$. Thus, the evaluation map $e: X \to \prod_{f \in C^*(X)} I_f$ is an embedding. THUS:

Thm: A space X is $T_{3\frac{1}{2}}$ iff it is homeomorphic to some subspace of some cube.