Products & Quotients

Let $\{X_\alpha\}_{\alpha\in I}$ be a set of top. spaces. The cartesian product of the sets is $X = \prod_{\alpha\in I} X_\alpha = \{f : f$ is a function w/ $\mathrm{dom}(f) = I$ and $\forall \alpha\in I,\ f(\alpha)\in X_\alpha$

[for finite collections, these are ordered tuples. For ctble collections sequences etc.

AC is equiv. to the statement: An arbitrary product of nonempty sets is nonempty. For $\alpha\in I$, define the α^{th} projection map $\Pi_\alpha : X \to X_\alpha$ by $\Pi_\alpha(f) = f_\alpha$.

Def^n: A base for the (Tychonoff) product topology on $X = \prod_{\alpha\in I} X_\alpha$ is given by $\mathcal{B} = \{\bigcap_{i=1}^{n} \Pi_{\alpha_i}^{-1}(U_{\alpha_i}) : n\in\omega,\ U_{\alpha_i}\in\tau_{\alpha_i}$ for each $\alpha_i,\ i=1,\dots,n\}$.

Pf this is a base: For any $\alpha\in I$, $X_\alpha\in\tau_\alpha$. Thus, $\Pi_\alpha^{-1}(X_\alpha) = \{f : f(\alpha)\in X_\alpha\} = X$. So $\bigcup\mathcal{B} = X$. Less stupidly, suppose $B_1, B_2\in\mathcal{B}$. Then $B_1 = \Pi_{\alpha_1}^{-1}(U_{\alpha_1})\cap\dots\cap\Pi_{\alpha_n}^{-1}(U_{\alpha_n})$, $B_2 = \Pi_{\beta_1}^{-1}(V_{\beta_1})\cap\dots\cap\Pi_{\beta_n}^{-1}(V_{\beta_n})$

Note that by reindexing, WLOG we can write

$B_1 = \Pi_{\gamma_1}^{-1}(U_{\gamma_1})\cap\Pi_{\gamma_2}^{-1}(U_{\gamma_2})\cap\dots\cap\Pi_{\gamma_N}^{-1}(U_{\gamma_N})$

$B_2 = \Pi_{\gamma_1}^{-1}(V_{\gamma_1})\cap\dots\cap\Pi_{\gamma_N}^{-1}(V_{\gamma_N})$. [Since if $\beta_i\neq\alpha_j$ for some i,j, $B_1 = B_1\cap\Pi_{\beta_i}^{-1}(X_{\beta_i})$]. Then $B_1\cap B_2 = \Pi_{\gamma_1}^{-1}(U_{\gamma_1}\cap V_{\gamma_1})\cap\dots\cap\Pi_{\gamma_N}^{-1}(U_{\gamma_N}\cap V_{\gamma_N})$

$\in\mathcal{B}$. So $\mathcal{B}$ is a base. $\boxed{\text{}}$

Alternatively: the product topology can be defined by a subbase $\mathcal{S} = \{\Pi_\alpha^{-1}(U_\alpha) : \alpha\in I,\ U_\alpha\in\tau_\alpha\}$. Compare this to the box topology which has as a base $\mathcal{B}_{box} = \{\prod_{\alpha\in I} U_\alpha : U_\alpha\in\tau_\alpha$ for each $\alpha\in I\}$

In general, the box topology is finer than the product topology.

If $|I| < \omega$, then the box topology equals the product topology,

since $U \in \mathcal{B}_{box}$ iff $U = \prod_{i=1}^{n} U_i = \{(x_1, x_2, \dots, x_n) : x_i \in U_i, i=1,\dots,n\}$

$= \bigcap_{i=1}^{n} \{\vec{x} : x_i \in U_i\} = \bigcap_{i=1}^{n} \{\vec{x} : \pi_i(\vec{x}) \in U_i\} = \bigcap_{i=1}^{n} \pi_i^{-1}(U_i) \in \mathcal{B}_{tychonoff}$.

Examples/Specific Facts:

- The product topology $\mathbb{R}_{std} \times \mathbb{R}_{std}$ is homeomorphic to $\mathbb{R}^2_{std}$

In fact, they're the same topology. If $U = B_{\rho_{std}}(x, \varepsilon_x) \times B_{\rho_{std}}(y, \varepsilon_y)$,

then letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, $B_{\rho_{std}}((x,y), \varepsilon) \subseteq U$. So $\tau_{std} \subseteq \tau_{product}$.

Similarly, if $B_{\rho_{std}}((x,y), \varepsilon)$ is an open ball in $\mathbb{R}_2^{std}$, then $\exists \varepsilon_{square} > 0$

st $B_{\rho_{std}}(x, \varepsilon_{square}) \times B_{\rho_{std}}(y, \varepsilon_{square}) \subseteq B_{\rho_{std}}((x,y), \varepsilon)$, so $\tau_{product} \subseteq \tau_{std}$.

- Let $X = \mathbb{R}^{\mathbb{R}}$. Then the product top. on X is a topology on the set of real valued functions $f : \mathbb{R} \to \mathbb{R}$.

[Note that if the B_α are restricted to being basic open sets in X_α, then $B = \{\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(B_{\alpha_i}) : n \in \omega, B_{\alpha_i} \in \mathcal{B}_{X_\alpha}\}$ is still a basis for the same topology, since any $U_\alpha \in \tau_\alpha$ can be written $\bigcup_{\beta \in J} B_\beta = U_\alpha$

so $\pi_{\alpha_i}^{-1}(U_{\alpha_i}) = \pi_{\alpha_i}^{-1}\left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{\beta \in J} \pi_{\alpha_i}^{-1}(B_\beta)$, a union of basic open sets in $\mathcal{B}$.] Thus, basic open sets in $\mathbb{R}^{\mathbb{R}}$ can be written in the form

$$U = \pi_{\alpha_1}^{-1}(B_{\rho_{std}}(x_1, \varepsilon_1)) \cap \dots \cap \pi_{\alpha_n}^{-1}(B_{\rho_{std}}(x_n, \varepsilon_n))$$

$$= \{g : \mathbb{R} \to \mathbb{R} : |g(x_i) - f(x_i)| < \varepsilon_i \text{ for each } i = 1, \dots, n\}$$

Thus, functions belonging to open sets in $\mathbb{R}^{\mathbb{R}}$ only need to look similar at a finite # of pts.

[In the box topology, we can define open sets consisting of functions which are uniformally similar]

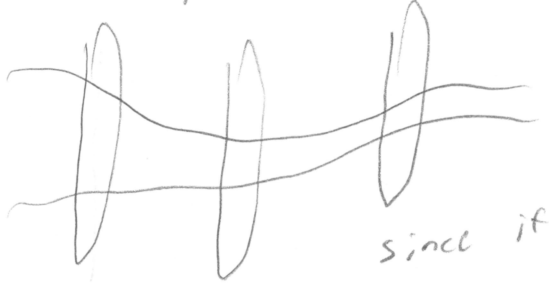
- Let S^1 denote the unit circle as a subspace of $\mathbb{R}^2_{std}$. and $I = [0,1]_{std}$. Then $S^1 \times I$ is called a <u>cylinder</u>, and $S^1 \times S^1$ is called a <u>torus</u>.

Thm: For $\beta \in I$, $\Pi_\beta : X \to X_\beta$ is a cts open mapping, but not necessarily closed and certainly not injective. (but certainly is surjective)

Pf: That Π_β is cts follows from the def^n: If $U_\alpha \in \tau_\alpha$, then $\Pi_\beta^{-1}(U_\alpha)$ is by def^n a basic open set in X. If B is basic open in X, then $B = \Pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \Pi_{\alpha_n}^{-1}(U_{\alpha_n})$

Then $\Pi_\beta(B) = \Pi_\beta(\Pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \Pi_{\alpha_n}^{-1}(U_{\alpha_n}))$
$= \Pi_\beta(\Pi_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \dots \cap \Pi_\beta(\Pi_{\alpha_n}^{-1}(U_{\alpha_n}))$

$\Pi_\beta(\Pi_{\alpha_i}^{-1}(U_{\alpha_i})) = X_\beta$ if $\alpha_i \neq \beta$, since $\Pi_{\alpha_i}^{-1}(U_{\alpha_i})$ contains functions through every point in X_β, and



$= U_{\alpha_i}$ if $\beta = \alpha_i$ (Clearly Π_β is surjective since if $x \in X_\beta$, then any function f st $f(\beta) = x$ will map to x. Thus $\Pi_\beta(\Pi_\beta^{-1}(U_{\alpha_i})) = U_{\alpha_i}$). In either case, it's open. So $\Pi(B)$ is a finite intersection of open sets, ie open. If U is open in X, then $U = \bigcup_{\gamma \in K} B_\gamma$, where $B_\gamma \in \mathcal{B}$ for each γ, so $\Pi(U) = \Pi(\bigcup_{\gamma \in K} B_\gamma) = \bigcup_{\gamma \in K} \Pi(B_\gamma)$, a union of open sets, ie open. Clearly Π_β is never injective as long as $|I| > 1$, since if $\Pi_\beta(f) = x$, then define $g(\delta) \neq f(\delta)$, $g(\beta) = x$, then $\Pi_\beta(f) = \Pi_\beta(g)$ but $f \neq g$.

As a counterexample to π_β being a closed map, consider $X = \mathbb{R}^2_{std}$, let $F = \{(x,y) : x > 0 \wedge y = \frac{1}{x}\}$.

F is closed in $\mathbb{R}^2_{std}$ by the closed graph thm (prove)

But $\pi_1(F) = (0, \infty)$, an open set in $\mathbb{R}_{std}$. $\boxtimes$

($\circledast_1$)

Thm: The product topology is the weakest topology on $\prod_{\alpha \in I} X_\alpha$ st π_β is cts for each $\beta \in I$. [weakest in the sense that if τ is any other topology under which our hypothesis is true, then τ is finer than the product topology, ie $\tau_{prod} \subseteq \tau$]

Pf: Suppose τ is a topology on $\prod X_\alpha$ st π_β is cts for each $\beta \in I$. Then if $U_\beta \in \tau_\beta$, $\pi_\beta^{-1}(U_\beta) \in \tau$. Recall that $\mathcal{S} = \{\pi_\alpha^{-1}(U_\alpha) : \alpha \in I, U_\alpha \in \tau_\alpha\}$ is a subbase for τ_{prod}. Thus every element of $\mathcal{S}$ is open in τ. Thus, if $U \in \tau_{prod}$, then U is a union of finite intersections of elts of τ, ie $U \in \tau$. So τ is finer than τ_{prod}. $\boxtimes$

($\circledast_2$)

Thm: A map $f : X \to \prod_{\alpha \in I} X_\alpha$ is cts iff $\pi_\alpha \circ f : X \to X_\alpha$ is cts for each $\alpha \in I$

Pf: The composition of cts functions is cts, so if $\pi_\alpha \circ f$ is not cts, then either π_α or f are not cts. But we showed that π_α has to be cts, so f is our culprit, revealing the $\Rightarrow$ direction contrapositively.
Conversely, if $\pi_\alpha \circ f$ is cts. If $\pi_\alpha^{-1}(U_\alpha)$ is a subbasic open set in $\prod_{\alpha \in I} X_\alpha$, then $f^{-1}(\pi^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) \in \tau_X$ by hypothesis.
Since $f^{-1}(\bigcap A_n) = \bigcap f^{-1}(A_n)$ and $f^{-1}(\bigcup A_\alpha) = \bigcup f^{-1}(A_\alpha)$, this is sufficient for $f^{-1}(U) \in \tau_X$ $\forall$ open $U \in \prod_{\alpha \in I} X_\alpha$. $\boxtimes$

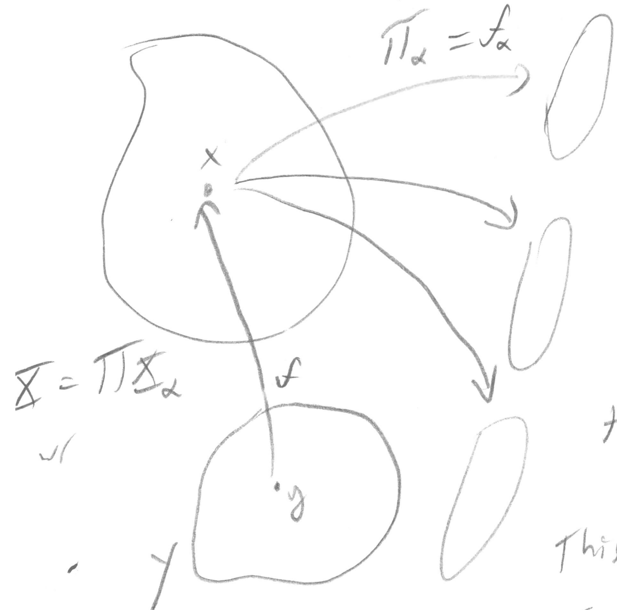
$\boxed{\begin{array}{l}\text{Thm: A countable product of } 1^{st} \text{ ctble spaces is } 1^{st} \text{ ctble.} \\ \text{A countable product of } 2^{nd} \text{ ctble spaces is } 2^{nd} \text{ ctble.} \\ \text{A countable product of seperable spaces is seperable.} \\ \text{A countable product of metric spaces is a metric space.}\end{array}}$

Pf (part 1): The 1^{st} and 2^{nd} ctble cases are trivial, since if $\{\mathcal{B}_n\}_{n\in\omega}$ is a ctble base for X_n, ie $\mathcal{B}_n = \{B_{n_k}\}_{k\in\omega}$ for each n, then a base for $\prod_{n=1}^{\infty} X_n$ is $\mathcal{B} = \{\pi_{n_1}^{-1}(B_{n_1}) \cap \dots \cap \pi_{n_m}^{-1}(B_{n_m}) : n, m \in \omega\}$

Clearly, $|\mathcal{B}| = |\omega \times \omega| = \omega$, so $\prod_{n\in\omega} X_n$ is 2^{nd} ctble. Further, if X_n is 1^{st} ctble for each n, then if $f \in \prod X_n$ then $\pi_n(f)$ has a nbhd base $\mathcal{B}_n = \{B_{n_m}\}_{m\in\omega}$ for each n. We claim again that $\mathcal{B}_f = \{\pi_{n_1}^{-1}(B_n) \cap \dots \cap \pi_{n_k}^{-1}(B_{n_k}) : n, k \in \omega\}$ is a nbhd base for f. Since $\pi_{n_i}(f) \in B_{n_m}$ for each n, m we "clearly" have that $f \in \pi_{n_1}^{-1}(B_{n_1}) \cap \dots \cap \pi_{n_k}^{-1}(B_{n_k})$, $\forall n, k \in \omega$. Further if U is an open set in $\prod X_\alpha$ containing f, then $\exists\, \pi_{n_1}^{-1}(U_{n_1}) \cap \dots \cap \pi_{n_m}^{-1}(U_{n_m})$ st $f \in \pi_{n_1}^{-1}(U_{n_1}) \cap \dots \cap \pi_{n_m}^{-1}(U_{n_m}) \subseteq U$, ie $\pi_{n_i}(f) \in U_{n_i}$ for each $i = 1, \dots, m$. Thus, since $\mathcal{B}_{n_i}$ is a nbhd base of $\pi_{n_i}(f)$, $\exists\, B_{n_{i_k}} \in \mathcal{B}_{n_i}$ st $\pi_{n_i}(f) \in B_{n_{i_k}} \subseteq U_{n_i} \Rightarrow f \in \pi_{n_1}^{-1}(B_{n_{1_k}}) \cap \dots \cap \pi_{n_m}^{-1}(B_{n_{m_k}}) \subseteq U$.

So $\prod X_n$ is 1^{st} ctble.

Seperable and metric space pfs forthcoming $\blacksquare$



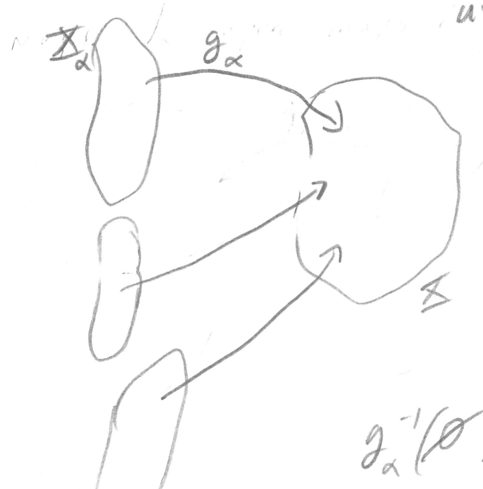
Thinking of X as just an arbitrary set and the f_α's as arbitrary functions $f_\alpha : X \to X_\alpha$.

(A_1) says that τ_{prod} is the weakest topology on X st if $V_\alpha \in \tau_\alpha$, then $f_\alpha^{-1}(V_\alpha) \in \tau_{prod}$.

This can be done naturally by simply defining all sets of the form $\{f_\alpha^{-1}(V_\alpha) : \alpha \in I, V_\alpha \in \tau_\alpha\}$ to be a subbase. It is reasonable to assume this would be weakest and it is. Further, if $f : Y \to X$ is some function from some space Y to X, then checking Y is cts equates by (A_2) to checking that each $f_\alpha \circ f : Y \to X_\alpha$ is cts for each α.

Dual to the weak topology induced on X by the f_α, we can define the strong topology in the following way. If $g_\alpha : X_\alpha \to X$ is a collection of functions from spaces X_α onto X (so now we're assuming that X is already a space and the X_α are just sets) then the strong topology induced on X_α by g_α is the finest topology on X st each g_α is cts. A somewhat natural way to attempt to construct such a topology on X is to define $\tau = \{U \subseteq X : g_\alpha^{-1}(U) \in \tau_\alpha \ \forall \alpha \in I\}$

ie U is open in τ iff $g_\alpha^{-1}(U)$ is open in each X_α.



Showing τ is a topology:

$g_\alpha^{-1}(\emptyset) = \emptyset \in \tau_\alpha \ \forall \alpha \in I$ so ✓, $g_\alpha^{-1}(X) = X_\alpha \in \tau_\alpha \ \forall \alpha \in I$ ✓

Let $\{U_\beta\}_{\beta \in J} \subseteq \tau$. Let $U = \bigcup_{\beta \in J} U_\beta$. Then $g_\alpha^{-1}(U) = g_\alpha^{-1}\left(\bigcup_{\beta \in J} U_\beta\right)$

$\overset{?}{=} \bigcup_{\beta \in J} g_\alpha^{-1}(U_\beta)$, a union of open sets in α since $g_\alpha^{-1}(U_\beta) \in \tau_\alpha$

$\forall \alpha \in I$. Thus $U \in \tau$. Finally, $U, V \in \tau \Rightarrow g_\alpha^{-1}(U \cap V) = g_\alpha^{-1}(U) \cap g_\alpha^{-1}(V)$

which is in τ_α. So τ is a topology.

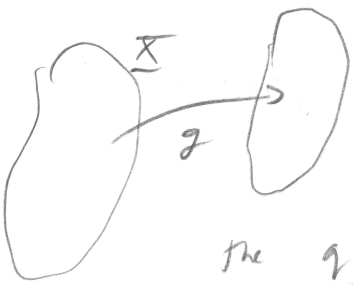
<u>Claim</u>: τ is the strongest topology st each $g_\alpha : X_\alpha \to X$ is cts $\forall \alpha \in I$.

<u>Pf</u>: Clearly by construction, g_α is cts wrt τ. Let τ' be another topology st g_α is cts for each α, and let $U \in \tau'$.

wts $U \in \tau$. $U \in \tau' \Rightarrow g_\alpha^{-1}(U) \in \tau_\alpha$ for each $\alpha \in I$

$\Rightarrow U \in \tau \Rightarrow \tau' \subseteq \tau$. [However, if $U \in \tau$ ie $g_\alpha^{-1}(U) \in \tau_\alpha$ $\forall \alpha \in I$, but there's nothing to suggest this is a sufficient condition for $U \in \tau'$! so U might not be in τ', so τ is definitely finer than τ' but τ' is not necessarily finer than τ].

Now let X be a space, Y be a set, and $g : X \to Y$ be an onto map. Then the <u>quotient topology induced on Y by g</u> (τ_g) is the strongest topology st g st $g^{-1}(U) \in \tau_X$ for each $U \in \tau_g$. The space (Y, τ_g) is called the <u>quotient space induced by X</u> and g itself is called a <u>quotient map</u>.



Ie g is a quotient map $\forall U \subseteq Y$, $U \in \tau_g$ iff $g^{-1}(U) \in \tau_X$. Some authors say g is "strongly cts"

Note $U \in \tau_g$ iff $g^{-1}(U) \in \tau_X$ iff $X - g^{-1}(U) \in \lambda_X$ iff

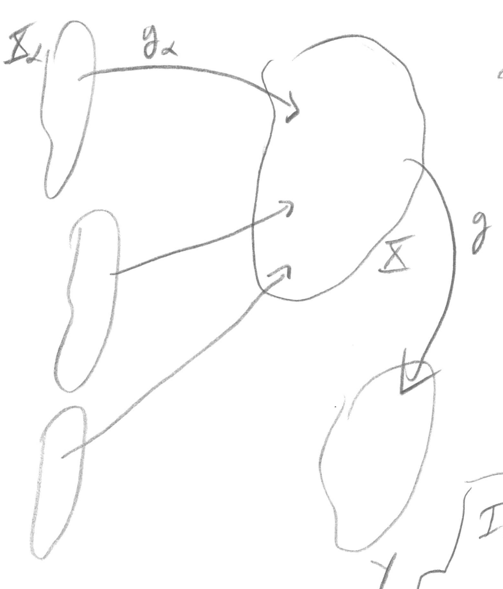
iff $g^{-1}(Y) - g^{-1}(U) \in \lambda_X$ iff $\boxed{g^{-1}(Y - U) \in \lambda_X \text{ iff } Y - U \in \lambda_Y}$

Alternative def^n of quotient map

Thm: If X, Y are top. spaces $f: X \to Y$ are cts and either open or closed, then the topology τ on Y is the quotient topology τ_f

Pf: Since g is already cts wrt τ_Y and τ_f is the strongest top. on Y st g is cts, already have $\tau_Y \subseteq \tau_f$. For the other direction, if $U \in \tau_f$, then by def^n of τ_f, $f^{-1}(U) \in \tau_X$. If f is an open map wrt τ_Y, then $f(f^{-1}(U)) \in \tau_Y$, but since [are we assuming that f is onto?] $f(f^{-1}(U)) = U$, we have that $U \in \tau_Y$, and so $\tau_f \subseteq \tau_Y$, ie $\tau_f = \tau_Y$.

If f is a closed map, then $U \in \tau_f \Rightarrow f^{-1}(U) \in \tau_X \Rightarrow f^{-1}(Y-U) \in \lambda_X \Rightarrow f(f^{-1}(Y-U)) \in \lambda_Y$ since f is a closed map, but (again I guess assuming f is onto?) $f(f^{-1}(Y-U)) = Y-U \in \lambda_Y$, so $U \in \tau_Y$, ie $\tau_f \subseteq \tau_Y$ ie $\tau_f = \tau_Y$. $\boxed{/\!/}$



Thm (The Dual of θ_2): Let $g_\alpha : X_\alpha \to X$ be a collection of functions from spaces (X_α, τ_α) to a space (X, τ), where τ is the strong topology induced on X by the g_α. Then for some other space (Y, τ_Y), a function $g: X \to Y$ is cts iff $g \circ g_\alpha : X_\alpha \to Y$ is cts for each $\alpha \in I$.

In particular, if $f: X \to Y$ is a quotient map of X onto Y with the quotient topology, and $g: Y \to Z$ is some function into the space (Z, τ_Z), then $g: Y \to Z$ is cts iff $g \circ f: X \to Z$ is cts.

Pf: If g is cts, then since f is a quotient map and thus also cts, the composition $g \circ f$ must be cts.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g \circ f"'] & Y \arrow[dl, "g"] \\ Z & \end{tikzcd}$$

Conversely, if $g \circ f$ is cts, then let $U \in \mathcal{T}_Z$. Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$, so since f is a quotient map, $g^{-1}(U) \in \mathcal{T}_Y$ iff $f^{-1}(g^{-1}(U)) \in \mathcal{T}_X \Rightarrow g^{-1}(U) \in \mathcal{T}_Y$. Thus, g is cts. $\boxed{/\!/}$

Next, let $\mathcal{D}$ be a partition of X (Willard calls it a decomposition) a space

we can define a topology $\mathcal{T}_{\mathcal{D}}$ on $\mathcal{D}$ called the decomposition space of X as follows: $\mathcal{U} \subseteq \mathcal{D}$ is open iff $\bigcup\{D : D \in \mathcal{U}\} \in \mathcal{T}_X$

[Note $\bigcup\{D : D \in \mathcal{U}\} = \bigcup\{[x] : [x] \in \mathcal{U}\}$].

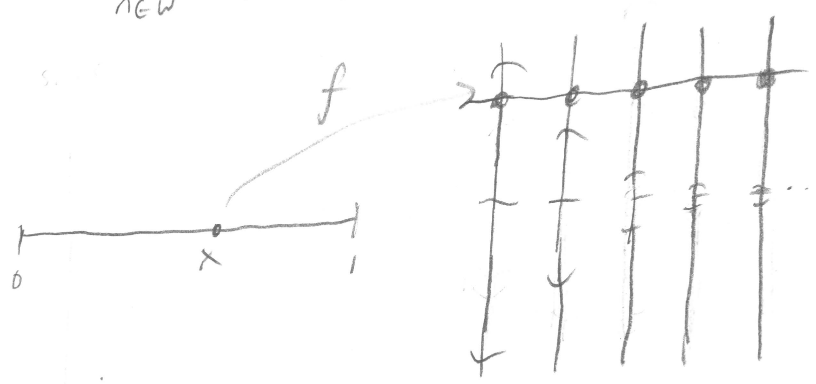
Pf this is a Topology: $\bigcup \emptyset = \emptyset$ and $\bigcup \mathcal{D} = X$ since $\mathcal{D}$ is a partition, so ✓. If $\{\mathcal{U}_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}_{\mathcal{D}}$, let $\mathcal{U} = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$. Then $\bigcup\{D : D \in \mathcal{U}_\alpha\} = \underset{\mathcal{T}_X}{\overset{\cap}{U_\alpha^*}}$ for each $\alpha \in I \Rightarrow \bigcup\{D : D \in \bigcup_{\alpha \in I} \mathcal{U}_\alpha\} = \bigcup_{\alpha \in I} \bigcup\{D : D \in \mathcal{U}_\alpha\}$

$$= \bigcup U_\alpha^* \in \mathcal{T}_X \ \checkmark$$

If $\mathcal{U}, \mathcal{V} \in \mathcal{T}_{\mathcal{D}}$, then $\bigcup\{D : D \in \mathcal{U} \cap \mathcal{V}\} = \bigcup\{D : D \in \mathcal{U}\} \cap \{D : D \in \mathcal{V}\}$

An example of τ_{box} vs τ_{prod}

Let $f:\mathbb{R}\to\mathbb{R}^\omega$ be given by $f(x)=(x,x,x,x,\dots)$ and consider $U=\prod_{n\in\omega}\left(-\frac{1}{2^n},\frac{1}{2^n}\right)\subseteq\mathbb{R}^\omega$. Note that $U\in\tau_{box}$ clearly, but $U\notin\tau_{prod}$, (show)



Then $f^{-1}(U)=\{x\in\mathbb{R}: -\frac{1}{2}<x<\frac{1}{2} \wedge -\frac{1}{4}<x<\frac{1}{4} \wedge \dots\}$

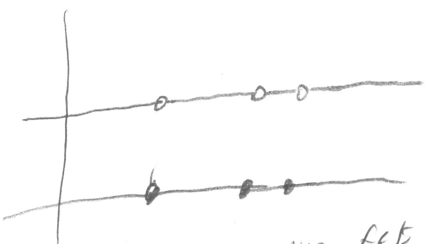
$=\bigcap_{n\in\omega}\left(-\frac{1}{2^n},\frac{1}{2^n}\right)=\{0\}\in\lambda_{\mathbb{R}_{std}}$

So the inverse image of the open set U in τ_{box} is closed, ie f is not cts. However, for each $n\in\mathbb{N}$, $\pi_n\circ f=\pi_n(f(x))=x$ is clearly a cts function from $\mathbb{R}_{std}$ to $\mathbb{R}_{std}$. Thus, in the box topology, functions might not be cts even if they look cts "in each coordinate".

[Why not in τ_{prod}? If it were, then since $f(x)=(0,0,0,\dots)\in U$, $\exists$ a basic open set of f $B=\pi_{n_1}^{-1}(U_{n_1})\cap\dots\cap\pi_{n_m}^{-1}(U_{n_m})$ st $f\in B\subseteq U$. Let $N=\max\{n_i\}_{i=1}^m$. Then since $B=B\cap\pi_{N+1}^{-1}(\mathbb{R}_{std})$, let $g\in\mathbb{R}^\omega$ be defined by letting $\pi_{n_i}(g)\in U_{n_i}$ for each $i=1,\dots,m$ but $\pi_M(g)=\frac{1}{2^N}$ $\forall M\geq N+1$. Then $g\in B$ but $g\notin U$, ※.]

<u>Moral</u>: The product topology is the only topology on $\prod_{\alpha\in I}X_\alpha$ st ctivity of functions in the topology is completely determined by whether or not they are cts functions from I into each coordinate.

Ex: We will show that $\mathbb{R}^{\mathbb{R}}$ is not 2^{nd} ctble. Recall that if $\mathbb{R}^{\mathbb{R}}$ is 1^{st} ctble, then $F \subseteq \mathbb{R}^{\mathbb{R}}$ is closed iff $\forall$ sequences $\{x_n\} \subseteq F$ st $x_n \to x$, $x \in F$. Let $E = \{ f \in \mathbb{R}^{\mathbb{R}} : f(x) = 0 \text{ or } 1 \text{ and } f(x) = 0 \text{ finitely often} \}$.



ex of a function $f \in E$

Let $g(x) = 0$ the constant function. Let $U(g)$ be a basic nbhd of g. Clearly, $g \notin E$. Let U ... Then $U(g) = \{ h \in \mathbb{R}^{\mathbb{R}} : |h(y) - g(y)| < \varepsilon \text{ if } y \in F \}$.

[Since $U(g) = \pi_{n_1}^{-1}(B(0, \varepsilon_1)) \cap \dots \cap \pi_{n_m}^{-1}(B(0, \varepsilon_m))$, let $\varepsilon = \min\{\varepsilon_i\}$ and $F = \{n_1, \dots, n_m\}$. WLOG $W = \pi_{F_1}^{-1}(B(0, \varepsilon)) \cap \dots \cap \pi_{F_m}^{-1}(B(0, \varepsilon))$].

Let $h(x) = \begin{cases} 0 & \text{if } x \in F \\ 1 & \text{else} \end{cases}$ Then $h \in E$ and $h \in U$, so $E \cap U \neq \emptyset$, ie $g \in \overline{E}$. Let $\{f_n\}_{n \in \omega}$ be any sequence in E. Then for each n, there exists a finite set A_n st $f_n(x) = \begin{cases} 0 & \text{if } x \in A_n \\ 1 & \text{else} \end{cases}$

Suppose $f_n \to f$ for some f. Then for any finite set $F \subseteq \mathbb{R}$, f_n is eventually in U.

$U(f) = \{ h \in \mathbb{R}^{\mathbb{R}} : |h(y) - f(y)| < \frac{1}{2} \text{ if } y \in F \}$.

That is, if $y \in F$, $\exists N$ st $\forall n \geq N$, $|f_n(y) - f(y)| < \frac{1}{2}$. If there is even a single $m \geq n$ st $F \not\subseteq A_m$, then for that $y \in F$ which is not in A_m, $f_n(y) = 1$, so $|f_n(y) - f(y)| = |1 - f(y)| < \frac{1}{2} \Rightarrow f(y) \neq 0$.

So the only way $f(y) = 0$ is if y is eventually in $\{A_n\}$. Thus, f can only be 0 at most at ctbly many pts since at best, A_n will eventually become an increasing chain (ie $\exists m$ st $A_m \subseteq A_{m+1} \subseteq \dots$) and $|A_n| \to \omega$. Thus f cannot equal g, which is 0 at unctbly many pts.

To summarize, g is an elt of $\mathbb{R}^{\mathbb{R}}$ which is in the closure of a set E, but there is no sequence $\{f_n\} \subseteq E$ which converges to it.

This proves:

- $\mathbb{R}^{\mathbb{R}}$ is not 1^{st} ctble (contrapositively) and also not 2^{nd} ctble (also contrapositively)
- An uncountable product of 1^{st} ctble spaces need not be 1^{st} ctble
- An unctble product of 2^{nd} ctble spaces need not be 2^{nd} ctble.
- If X is not 1^{st} ctble, it need not be the case that $x \in \bar{A} \Rightarrow \exists \{x_n\} \subseteq A$ st $x_n \to x$.

Note also that in this example, $g \in \bar{E} \Rightarrow g \in$

Another (simpler) ex: $[0, \omega_1]$. $[0, \omega_1) \subseteq [0, \omega_1]$ and $\overline{[0, \omega_1)} = [0, \omega_1]$ (for any open U st $\omega_1 \in U$, $\exists (\gamma, \omega_1]$ st $\omega_1 \in (\gamma, \omega_1] \subseteq U$. but $\gamma < \omega_1$ so $\gamma \in [0, \omega_1) \Rightarrow (\gamma, \omega_1] \cap [0, \omega_1) \neq \emptyset \Rightarrow U \cap [0, \omega_1) \neq \emptyset \Rightarrow \omega_1 \in \overline{[0, \omega_1)} \Rightarrow \overline{[0, \omega_1)} = [0, \omega_1]$.)

A sequence in $[0, \omega_1)$ is a sequence of countable ordinals and there does not exist a sequence of ctble ordinals converging to ω_1 since the cofinality of any such sequence is ω. $\boxed{\checkmark}$

More on Quotient Maps

Def^n: If $\mathfrak{X}$ is a top. space, Y a set, and $g: \mathfrak{X} \to Y$ is an onto map, then the collection $\tau_Y = \{G \subseteq Y : g^{-1}(G) \in \tau_{\mathfrak{X}}\}$ is a topology on Y, called the quotient topology on Y induced by g. Y is called a quotient space of $\mathfrak{X}$, and g is called a quotient map.

Thm: If $\mathfrak{X}, Y$ are spaces, $f: \mathfrak{X} \to Y$ is cts and either open or closed, then the topology on Y is the quotient topology on Y induced by f.

Def^n Let $\mathcal{D}$ be a partition of a space $(\mathfrak{X}, \tau)$. Define $\tau_{\mathfrak{X}/\sim}$ on $\mathfrak{X}/\sim$ by U open iff $\bigcup\{[x] : [x] \in U\} \in \tau_{\mathfrak{X}}$.

Then this is a topology on $\mathfrak{X}$.

Thm: Let $(\mathfrak{X}, \tau)$ a space, let Y be the quotient space induced by some onto map $f: \mathfrak{X} \to Y$. Then $\exists$ an equivalence rln $\sim$ on $\mathfrak{X}$ st $\mathfrak{X}/\sim$ is homeomorphic to Y.

Thus, we may think of all quotient spaces as being defined by an equivalence rln.

Def^n: If $A \subseteq \mathfrak{X}$, we say A is saturated wrt $\sim$ if $\forall x, y \in \mathfrak{X}$, $x \in A$ and $x \sim y \Rightarrow y \in A$. (So A is really a union of equivalence clns.

quotient maps $q : X \to X/\sim$ are always cts/onto

Let $A \subseteq X$, and $sat(A) := \{x : \exists y \in A : x \sim y\}$

(ie the union of all equivalence classes of elts of A)

Thm: $q : X \to X/\sim$ is open iff $\forall U \in \tau_X$, $sat(U) \in \tau_X$

$q : X \to X/\sim$ is closed iff $\forall F \in \lambda_X$, $sat(F) \in \lambda_X$.