Mostly Metric Space Stuff

The trivial topology on $\mathbb{X}$ is $\tau = \{\emptyset, \mathbb{X}\}$. That this is in fact a topology is trivial.

The discrete topology on $\mathbb{X}$ is $\tau = \mathcal{P}(\mathbb{X})$. Also trivially a topology.

A metric ρ on a set $\mathbb{X}$ is a function $\rho : \mathbb{X}^2 \to \mathbb{R}$ st

1) $\rho(x,y) \geq 0$ and $\rho(x,y) = 0 \iff x = y$ (positive semidefinite)

2) $\rho(x,y) = \rho(y,x)$ (symmetric)

3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ (triangle inequality)

(weakening condition 1) to $\rho(x,y) = 0 \Leftarrow x = y$ defines a pseudometric)

Thm: Define $B_\rho(x,\varepsilon) = \{y \in \mathbb{X} : \rho(x,y) < \varepsilon\}$ to be the open ε-ball about x wrt ρ. Define τ_ρ by $U \in \tau_\rho$ iff $\forall x \in U, \exists \varepsilon > 0$ st $B_\rho(x,\varepsilon) \subseteq U$. Then τ_ρ is a topology.

pf: Let $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_\rho$. Let $x \in U = \bigcup_{\alpha \in I} U_\alpha$. Then $x \in U_\alpha$ for some fixed $\alpha \in I$, so $U_\alpha \in \tau_\rho \Rightarrow \exists \varepsilon > 0$ st $B_\rho(x,\varepsilon) \subseteq U_\alpha$ ie $B_\rho(x,\varepsilon) \subseteq U$ ie $U \in \tau$. $\emptyset \in \tau_\rho$ vacuously. Let $x \in \mathbb{X}$. Then $\mathbb{X} = \bigcup_{n \in \mathbb{N}} B_\rho(x,n)$ so $\mathbb{X} \in \tau_\rho$. If $U, V \in \tau_\rho$, $U \cap V \neq \emptyset$, then $\exists x \in U \cap V$, $x \in U \Rightarrow \exists \varepsilon_1 > 0$ st $B_\rho(x,\varepsilon_1) \subseteq U$, $x \in V \Rightarrow \exists \varepsilon_2 > 0$ st $B_\rho(x,\varepsilon_2) \subseteq V$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\rho(x,\varepsilon) \subseteq B_\rho(x,\varepsilon_1) \subseteq U$, and similarly $B_\rho(x,\varepsilon) \subseteq V$, so $B_\rho(x,\varepsilon) \subseteq U \cap V$ ie $U \cap V \in \tau_\rho$. $\boxtimes$

Note, any function $f : \mathbb{X}^2 \to \mathbb{R}$ defines a topology τ_f by this same argument! We didn't use any properties of ρ for this proof.

Alternatively, note that for each $x \in X$, $\xi_x := \{B_\rho(x,\varepsilon) : \varepsilon > 0\}$

Then $B_\rho(x,\varepsilon_1) \cap B_\rho(x,\varepsilon_2) = B_\rho(x, \min\{\varepsilon_1, \varepsilon_2\}) \in \xi_x$, so

This defines a filter base at each pt for τ_ρ. This begs the question, when is ξ_x a nbhd base?

Let $y \in B_\rho(x,\varepsilon)$, so $\rho(x,y) < \varepsilon$. Let $\delta = \varepsilon - \rho(x,y)$

If $z \in B_\rho(y,\delta)$, then assuming the triangle inequality,

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z) < \rho(x,y) + \delta = \rho(x,y) + (\varepsilon - \rho(x,y)) = \varepsilon.$$

So $z \in B_\rho(x,\varepsilon) \Rightarrow B_\rho(y,\delta) \subseteq B_\rho(y,\varepsilon)$.

$\therefore$ If our function satisfies the triangle inequality then ξ_x will always be a nbhd base. The space (X, τ_ρ) is the metric space induced by τ_ρ.

If (X, τ) is a space st $\exists$ a metric ρ where $\tau_\rho = \tau$, then we say that (X,τ) is metrizable.

The standard metric on $\mathbb{R}$ is given $\rho(x,y) = |x-y|$. $\mathbb{R}_{std}$ is what denotes this metric space.

Two metrics are equivalent if they induce the same topology

Thm: Every metric ρ is equivalent to a bdd metric ρ'

(bdd $\Rightarrow \forall x,y \in X$, $\rho(x,y) \leq M$ for some $M \in \mathbb{R}$)

Pf: Define $\rho'(x,y) = \min\{\rho(x,y), 1\}$. Clearly, $\rho'(x,y) = \rho'(y,x)$, and $\rho'(x,y) \geq 0$ with equality iff $x=y$. Let $x,y,z \in X$. If $\rho(x,y)$ or $\rho(y,z) \geq 1$, then $\rho(x,z) \leq 1$ clearly, and then WLOG $\rho(x,y) + \rho(y,z) > 1$

so $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Else, $\rho(x,y), \rho(x,z) \leq 1$

$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ since ρ is a metric, and

$\rho'(x,z) \leq \rho(x,z) \leq \rho(x,y) + \rho(y,z) = \rho'(x,y) + \rho'(y,z)$. So ρ'

satisfies the triangle inequality and is thus a metric.

Now we claim $\mathcal{T}_\rho = \mathcal{T}_{\rho'}$. Note that since ρ is a metric and

so is ρ', the collection $\mathcal{B}_x = \{B_\rho(x,\varepsilon) : \varepsilon > 0\}$ is a nbhd base

(same for ρ'.) Let $B_\rho(x,\varepsilon)$ be a basic nbhd of x. Then $\forall y \in B_\rho(x,\varepsilon)$,

$\rho(x,y) \leq \varepsilon$, but $\rho'(x,y) \leq \rho(x,y)$, so $B_{\rho'}(x,y) \subseteq B_\rho(x,y)$. Thus

by the Hausdorff criterion (p.38 of Willard), $\mathcal{T}_{\rho'} \subseteq \mathcal{T}_\rho$. Next,

let $B_{\rho'}(x,\varepsilon)$ be a basic nbhd of x. Let $\delta = \min\{\varepsilon, 1\}$.

Then consider $B_\rho(x,\delta)$. If $y \in B_\rho(x,\delta)$ then $\rho(x,y) < \delta$ and $< \varepsilon$.

so $B_\rho(x,\delta) \subseteq B_{\rho'}(x,\varepsilon) \Rightarrow \mathcal{T}_\rho \subseteq \mathcal{T}_{\rho'}$ again by the same criterion.

Therefore $\mathcal{T}_\rho = \mathcal{T}_{\rho'}$. $\boxed{\checkmark}$

Note that ρ for $\mathbb{R}_{std}$ is unbdd: for any $M \in \mathbb{R}$, let $y = x + M + 1$

then $|x - y| = |x - x + M + 1| = M + 1 > M$ (assuming $x, y > 0$).

Thus metrics are not uniquely defined by a topology.

Note: If ρ is a metric, the function $\rho'(x,y) = \dfrac{\rho(x,y)}{\rho(x,y)+1}$ is also

an equivalent bounded metric. (pf is in HW2 #1).

Let $n \in \mathbb{N}$. The <u>standard Euclidean Space</u> $\mathbb{R}^n_{std}$ is induced by

the standard euclidean metric $\rho(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

[If $n=1$, $\mathbb{R}^n_{std} = \mathbb{R}_{std}$]

The Discrete Topology is Metrizable: Consider the discrete metric

$\rho(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{otherwise} \end{cases}$ Clearly (1) and (2) are satisfied, and if $x,y,z \in X$, $x \neq y$ or $y \neq z$

$\Rightarrow \rho(x,y) + \rho(y,z) \geq 1$, so $\rho(x,z) \leq 1 \Rightarrow \rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

All sets are open in $\mathcal{P}(X) = \mathcal{T}$. This is true iff $\forall x \in X$, $\{x\}$ is open [$\Rightarrow$ is trivial. $\Leftarrow$ if $U \in \mathcal{P}(X)$ then $U = \bigcup_{x \in U} \{x\} \in \mathcal{T}$]

But if $0 \leq \varepsilon < 1$, then $\forall y \in B_\rho(x,\varepsilon)$, $\rho(x,y) < 1 \Rightarrow \rho(x,y) = 0 \Rightarrow x = y$.

So $B_\rho(x,\varepsilon) = \{x\} \in \mathcal{T}_\rho$. So $\mathcal{T}_\rho = \mathcal{P}(X)$. $\boxed{\checkmark}$

The Trivial Topology is pseudometrizable (ie induced by a pseudometric)

Let ρ be the trivial pseudometric $\rho(x,y) = 0$ $\forall x,y \in X$.

$\mathcal{S}$ X a set and $\mathcal{T} = \{\emptyset, X\}$. Clearly this is a pseudometric: (2) and (3) are always true, and $x = y \Rightarrow \rho(x,y) = 0$ but if X contains more than one element then $\exists x,y \in X$ st $x \neq y$ so $\rho(x,y) = 0$ but $x \neq y$ ie ρ is a metric iff X contains only a single pt. If $x \in X$, then for any $\varepsilon \geq 0$, $B_\rho(x,\varepsilon) = X$, so $\mathcal{T}_\rho = \{\emptyset, X\}$.

Let $(X, \mathcal{T}_X)$ be a top. space and $Y \subseteq X$. Then the subspace top on Y is given by $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}_X\}$. The open sets in $\mathcal{T}_Y$ are called relatively open

Pf that this is a Top.: $\emptyset = \emptyset \cap Y$, so $\emptyset \in \mathcal{T}_Y$. $Y = X \cap Y$, so $Y \in \mathcal{T}_Y$. If $\{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{T}_Y$, then $U_\alpha = V_\alpha \cap Y$ for some $V_\alpha \in \mathcal{T}_X$

Thus $U = \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} V_\alpha \cap Y = \left(\bigcup_{\alpha \in I} V_\alpha\right) \cap Y \in \mathcal{T}_Y$ since $\bigcup_{\alpha \in I} V_\alpha \in \mathcal{T}_X$.

Finally, If $U, V \in \tau_Y$, then $U = A \cap Y$, $A \in \tau$, $V = B \cap Y$, $B \in \tau_{\mathbb{X}}$

so $U \cap V = (A \cap Y) \cap (B \cap Y) = (A \cap B) \cap Y \in \tau_Y$ since $A \cap B \in \tau_{\mathbb{X}}$

so τ_Y is a top. $\boxed{}$

Trivially, if ρ is a metric on $\mathbb{X}$, then $\rho \restriction_{Y \times Y}$ is a metric on Y.

If $(\mathbb{X}, \rho)$ is a metric space and $Y \subseteq \mathbb{X}$, then the Y-subspace is a metric space induced by $\rho \restriction_{Y \times Y}$.

Pf: If $U \in \tau_Y$, then $U = V \cap Y$ for some $V \in \tau_{\mathbb{X}}$

so since $\mathcal{B} = \{ B_\rho(x, \varepsilon) : x \in \mathbb{X}, \varepsilon > 0 \}$ is a basis

($\mathcal{B}_x = \{ B_\rho(x, \varepsilon) : \varepsilon > 0 \}$ is always a nbhd base at x).

$V = \bigcup_{\alpha \in I} B_\rho(x_\alpha, \varepsilon_\alpha)$ $\Rightarrow$ $U = \bigcup_{\alpha \in I} (B_\rho(x_\alpha, \varepsilon_\alpha) \cap Y)$

$= \bigcup_{\alpha \in I} B_{\rho \restriction_{Y \times Y}}(x_\alpha, \varepsilon_\alpha)$, so $\mathcal{B}_Y = \{ B_{\rho \restriction_{Y \times Y}}(x, \varepsilon) : x \in \mathbb{X}, \varepsilon > 0 \}$

is a basic for the Y subspace. $\boxed{}$

$\boxed{\text{Thm: All metric spaces are } 1^{st} \text{ ctble}}$

Pf: For any $x \in \mathbb{X}$, consider $\mathcal{B}_x = \{ B_\rho(x, q) : q \in \mathbb{Q} \}$. Clearly, this is ctble, and all $B_\rho(x, q)$ are nbhds of x. Also, if U is open w/ $x \in U$, then $U = \bigcup_{\alpha \in I} B_\rho(y_\alpha, \varepsilon_\alpha)$ for some $\{y_\alpha\}_{\alpha \in I} \subseteq \mathbb{X}$, $\{\varepsilon_\alpha\}_{\alpha \in I} \subseteq \mathbb{R}^+$. For some fixed, $y_\alpha = y$, $\varepsilon_\alpha = \varepsilon$, $x \in B_\rho(y, \varepsilon)$. Let $\delta = \rho(x, y) < \varepsilon$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (show this), $\exists q \in \mathbb{Q}$ st $q < \delta < \varepsilon$

so $B_\rho(x, q) \subseteq B_\rho(y, \varepsilon) \subseteq U$, thus this is a filter base for the nbhd filter, ie $\mathcal{B}_x$ is a ctble nbhd base. $\boxed{}$

[Dont have to use this just use archimedean property to pick $n \in \mathbb{N}$ st $\frac{1}{n} < \delta$, $\frac{1}{n} \in \mathbb{Q}$ clearly.]

$\boxed{\text{Thm: If } (X, \rho) \text{ is a metric space and is seperable, then it is } 2^{nd} \text{ ctble.}}$

Pf: Since (X, ρ) is seperable, it has a ctble dense set D.

ie $D = \{d_n\}_{n \in \mathbb{N}}$ is dense. Let $\mathcal{B} = \{B_\rho(d_n, q) : n \in \mathbb{N}, q \in \mathbb{Q}^+\}$.

Clearly $\mathcal{B}$ is countable. Let $U \in \tau$, $x \in U$. Let $\varepsilon > 0$ be

st $B_\rho(x, \varepsilon) \subseteq U$.

Since D is dense, $\exists n \in \mathbb{N}$ st $d_n \in B_\rho(x, \varepsilon)$. Let $\delta = \varepsilon - \rho(d_n, x) > 0$.

Choose $n \in \mathbb{N}$ st $\frac{1}{n} < \delta < \varepsilon$. Then $\forall y \in B_\rho(d_n, \frac{1}{n})$,

$$\rho(x, y) \leq \rho(y, d_n) + \rho(d_n, x) < \frac{1}{n} + \rho(d_n, x) < \delta + \rho(d_n, x)$$
$$= \varepsilon - \rho(d_n, x) + \rho(d_n, x) = \varepsilon.$$

$\therefore$ $B_\rho(d_n, \frac{1}{n}) \subseteq B_\rho(x, \varepsilon) \subseteq U$, and $B_\rho(d_n, \frac{1}{n}) \in \mathcal{B}$, so

$\mathcal{B}$ is a base for (X, ρ). $\blacksquare$

Corrollary: For metric spaces (X, ρ), X is always 1^{st} ctble, X is seperable.

and X is 2^{nd} ctble iff

Fact: If ρ, d are metrics on X and $\exists c \in \mathbb{R}$ st $\rho(x, y) \leq c\, d(x, y)$

$\forall x, y \in X$, then $\tau_\rho \subseteq \tau_d$.

Pf: Let $x \in X$, $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{c}$. (WLOG $c > 0$. If $c = 0$ then ρ is the trivial pseudometric). Then if $y \in B_d(x, \delta)$ then $d(x, y) < \frac{\varepsilon}{c}$,

but then $\rho(x, y) \leq c\, d(x, y) < c \frac{\varepsilon}{c} = \varepsilon \Rightarrow y \in B_\rho(x, \varepsilon)$.

So $B_d(x, \delta) \subseteq B_\rho(x, \varepsilon) \Rightarrow \tau_\rho \subseteq \tau_d$

Corollary: If additionally $\exists \frac{1}{b} \in \mathbb{R}$ st $d(x, y) \leq \frac{1}{b} \rho(x, y)$

ie $b\,d(x, y) \leq \rho(x, y) \leq c\,d(x, y)$, then $\tau_\rho = \tau_d$.

Thus, if $\exists b, d \in \mathbb{R}^+$ st $\forall x, y \in X$, $b\,d(x,y) \le \rho(x,y) \le c\,d(x,y)$, then ρ and d are equivalent metrics.

However, if $\mathcal{T}_d = \mathcal{T}_\rho$ this does not have to be the case.

If the boundedness condition is satisfied then we say that ρ and d are strongly equivalent.

Ex: $\rho_{std}(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$

$\rho_2(\vec{x},\vec{y}) = |x_1-y_1| + \dots + |x_n-y_n|$ taxi-cab metric

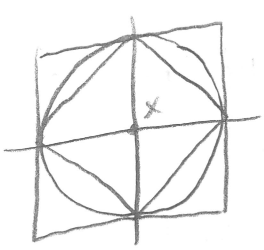
$\rho_3(\vec{x},\vec{y}) = \max\{|x_1-y_1|, \dots, |x_n-y_n|\}$ ℓ^∞ metric

Clearly, $\rho_3 \le \rho_2$ and ρ_{std}. Also,

$$\rho_{std}(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2} \le \sqrt{(x_1-y_1)^2} + \dots + \sqrt{(x_n-y_n)^2}$$
$$= |x_1-y_1| + \dots + |x_n-y_n| = \rho_2(\vec{x},\vec{y})$$

so $\rho_{std} \le \rho_2$. Finally, since $\rho_2(\vec{x},\vec{y}) \le n\,\rho_3(\vec{x},\vec{y})$

we have $\rho_3 \le \rho_{std} \le \rho_2 \le n\,\rho_3$, so these are all strongly equivalent, and thus induce the same topologies, namely $\mathbb{R}^n_{std}$.



$\leftarrow B_{\rho_2}(x,\varepsilon)$

$\leftarrow n B_{\rho_3}(x,\varepsilon)$

$\leftarrow B_{\rho_{std}}(x,\varepsilon)$.

Define $B_{\bar\rho}(x,\varepsilon) := \{y : \rho(x,y) \le \varepsilon\}$

Fact: $B_{\bar\rho}(x,\varepsilon)$ is closed.

Pf: Let $y \notin B_{\bar\rho}(x,\varepsilon)$. Then $\rho(x,y) > \varepsilon$.

Let $\delta = \rho(x,y) - \varepsilon > 0$, so $B_\rho(y,\delta) \cap B_{\bar\rho}(x,\varepsilon) = \emptyset$.

Thus, $B_\rho(y,\delta) \subseteq (B_{\bar\rho}(x,\varepsilon))^c$ ie $((B_{\bar\rho}(x,\varepsilon))^c)^\circ = (B_{\bar\rho}(x,\varepsilon))^c$

ie $(B_{\bar\rho}(x,\varepsilon))^c$ is open, ie $B_{\bar\rho}(x,\varepsilon)$ is closed. $\blacksquare$

Suppose

Alex Creiner
Topology HW 2

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1) Unfortunately $\frac{x}{1+x}$ is not convex on $[0,\infty)$, so I can't use the hint to make this easy

$\left(\frac{d^2}{d^2x}\left(\frac{x}{1+x}\right) = \frac{-2}{(1+x)^3} < 0 \ \forall x > 0 \right)$. I'm going to abbreviate

$\geq$ This is what convex means.

$\rho(x,y)$ and $\rho'(x,y)$ with ρ_{xy}, ρ'_{xy}. (I'm starting with the triangle inequality). Let $x,y,z \in X$.

Note that $\rho'_{xz} \leq \rho'_{xy} + \rho'_{yz}$

$$\text{iff } \frac{\rho_{xz}}{1+\rho_{xz}} \leq \frac{\rho_{xy}}{1+\rho_{xy}} + \frac{\rho_{yz}}{1+\rho_{yz}}$$

$$\text{iff } \rho_{xz} + \cancel{\rho_{xz}\rho_{yz}} + \cancel{\rho_{xz}\rho_{xy}} + \cancel{\rho_{xz}\rho_{xy}\rho_{yz}}$$
$$\leq \rho_{xy} + \cancel{\rho_{xy}\rho_{xz}} + \rho_{xy}\rho_{yz} + \cancel{\rho_{xy}\rho_{yz}\rho_{xz}} + \rho_{yz} + \cancel{\rho_{yz}\rho_{xz}} + \rho_{yz}\rho_{xy} + \rho_{yz}\rho_{xy}\rho_{xz}$$

(multiplying both sides by $(1+\rho_{xz})(1+\rho_{xy})(1+\rho_{yz})$ and distributing terms)

$$\text{iff } \rho_{xz} \leq \rho_{xy} + \rho_{yz} + 2\rho_{xy}\rho_{yz} + \rho_{yz}\rho_{xy}\rho_{xz}.$$

But since ρ is a metric,

$$\rho_{xz} \leq \rho_{xy} + \rho_{yz} \leq \rho_{xy} + \rho_{yz} + 2\rho_{xy}\rho_{yz} + \rho_{yz}\rho_{xy}\rho_{xz}$$

which we just showed was equivalent to $\rho'_{xz} \leq \rho'_{xy} + \rho'_{yz}$.

To finish off showing that ρ' is a metric, note that clearly $\rho_{xy} \geq 0 \Rightarrow \frac{\rho_{xy}}{1+\rho_{xy}} \geq 0$, so $\rho_{xy} \geq 0 \ \forall x,y \in X \Rightarrow \frac{\rho_{xy}}{1+\rho_{xy}} \geq 0 \ \forall x,y \in X$, and $(\rho_{xy} = 0 \Leftrightarrow x=y)$ iff $\left(\frac{\rho_{xy}}{1+\rho_{xy}} = 0 \Leftrightarrow x=y\right)$. Finally, if $\rho_{xy} = \rho_{yx}$, then $\frac{\rho_{xy}}{1+\rho_{xy}} = \frac{\rho_{yx}}{1+\rho_{yx}}$, so we're done. ∎

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3) For any metric space (X, ρ), every open set U is the increasing union of some countable sequence of closed sets F_n.

Pf: By question 2, we have that for all $\varepsilon > 0$, the set $B_\rho(U^c; \varepsilon) = \{x \in X : \rho(x, U^c) < \varepsilon\}$ is open. Thus The set $B_\rho(U^c; \varepsilon)^c = \{x \in X : \rho(x, U^c) \geq \varepsilon\}$ is closed. ✓

Define $F_n = B_\rho(U^c; \frac{1}{n})^c$ for each $n \in \mathbb{N}$. Clearly, this is an ascending sequence of sets, since $\frac{1}{n} > \frac{1}{n+1}$, so if $\rho(x, U^c) \geq \frac{1}{n}$ then surely $\rho(x, U^c) \geq \frac{1}{n+1}$. O.K. We now show $U = \bigcup_{n=1}^{\infty} F_n$. If $x \in U$, then since U is open there exists an $\varepsilon > 0$ so that $B_\rho(x, \varepsilon) \subseteq U$, ie $\rho(x, a) \geq \varepsilon$ $\forall a \in U^c$, ie $\rho(x, U^c) \geq \varepsilon$. ? Choose $n_0 \in \mathbb{N}$ so that $\frac{1}{n_0} < \varepsilon$, and then we see $\rho(x, U^c) > \frac{1}{n_0} \Rightarrow x \in F_{n_0}$, ie $x \in \bigcup F_n$, so $U \subseteq \bigcup_{n=1}^{\infty} F_n$. If $x \in \bigcup F_n$, then there exists a n_0 so that $\forall n \geq n_0$, $x \in F_n$, ie $\rho(x, U^c) \geq \frac{1}{n}$ $\forall n \geq n_0$, which would mean that for any $\varepsilon < \frac{1}{n}$, $B_\rho(x, \varepsilon) \subseteq U$, so $x \in U$.

Thus $U = \bigcup_{n=1}^{\infty} F_n$. ∎ O.K.

This clearly implies $x \notin U^c$, so $x \in U$.

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2) for $A \subseteq X$, $x, y \in X$, $A \neq \emptyset$ note that by the triangle inequality we have that for all $a \in A$,

$$\rho(y,a) \leq \rho(y,x) + \rho(x,a) = \rho(x,a) + \rho(x,y)$$

$$(*)$$

Thus, $\inf\limits_{a \in A}\{\rho(y,a)\} \leq \inf\limits_{a \in A}\{\rho(x,a) + \rho(x,y)\}$

$$= \inf_{a \in A}\{\rho(x,a)\} + \rho(x,y)$$

ie $\rho(y,A) \leq \rho(x,A) + \rho(x,y)$

don't use this notation, doesn't make sense.

[(*) If $x = \inf\limits_{a \in A}\{\rho(y,a)\}$ and $y = \inf\limits_{a \in A}\{\rho(x,a)\}$ with s t

$\rho(y,a) \leq \rho(x,a)$ $\forall a \in A$, then

$\overset{s}{x} \leq \rho(y,a) \leq \rho(x,a)$, ie $\overset{s}{\cancel{x}}$ is a lower bound for $\{\rho(x,a) : a \in A\}$. But since $\cancel{y}^{t}$ is the infimum of this set,

$\overset{s}{x} \leq \overset{t}{y}$, as desired]

with this known, we now show that $\forall \varepsilon > 0$, $B_\rho(A,\varepsilon) = \{x : \rho(x,A) < \varepsilon\}$ is open. Let $x \in B_\rho(A,\varepsilon)$. Then $\rho(x,A) < \varepsilon$, ie $\rho(x,A) = \alpha < \varepsilon$ for some $\alpha \geq 0$. Thus there is a $\delta > 0$ so that $\alpha + \delta < \varepsilon$. Consider $B_\rho(x,\delta)$ and note that if $y \in B_\rho(x,\delta)$, then by our inequality that we just showed,

$\rho(y,A) \leq \rho(x,A) + \rho(x,y) < \alpha + \delta < \varepsilon$, ie $y \in B_\rho(A,\varepsilon)$.

So $B_\rho(x,\delta) \subseteq B_\rho(A,\varepsilon)$, ie $B_\rho(A,\varepsilon)$ is open.

Likewise, for $x \in \{z : \rho(z,A) > \varepsilon\}$, $\rho(x,A) = \alpha > \varepsilon$ for some α, so there exists a $\delta > 0$ st $\alpha - \delta > \varepsilon$ still. Then for $y \in B_\rho(x,\delta)$ (continued on back)

2 continued) $\alpha \leq \rho(x, A) \leq \rho(y, A) + \rho(x, y)$

$$\Rightarrow \quad \rho(y, A) \geq \alpha - \rho(x, y) > \alpha - \delta > \varepsilon.$$

(since $\rho(x, y) < \delta$)

Thus $B_\rho(x, \delta) \subseteq \{z : \rho(z, A) > \varepsilon\}$, ie the set is open. Since the complement of an open set is closed, we have the other 2 claims for free. ☐

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