Ordinals and order spaces

A linear ordering $\leq$ is a binary rln on a set X which is reflexive, transitive, and
- $\forall x,y \in X,\ x \leq y \wedge y \leq x \Rightarrow x = y$
- $\forall x,y \in X,\ x \leq y \vee y \leq x$ (connected)

A linear ordering is a well order if $\forall S \in \mathcal{P}(X) - \{\emptyset\}$, S has a least element, ie an elt $x \in S$ st $\forall y \in S,\ y \geq x$.

An equivalent statement to AC: Every set can be well ordered.

More generally, a rln R on a set X is well-founded if $\forall S \in \mathcal{P}(X) - \{\emptyset\}$, $\exists x \in S$ st $\forall y \in S,\ \neg(yRx)$. So a well ordering is just a linear ordering which is well founded.

Dependant Choice. A weaker form of AC. It says: $\forall X \neq \emptyset$ and all entire binary relations X (entire means $\forall a \in X, \exists b \in X$ st aRb) there exists a sequence $\{x_n\}_{n \in \omega}$ st $x_n R x_{n+1}$ $\forall n \in \omega$.

(Dependant choice is also weaker than the axiom of ctble choice)

Dependant Choice is insufficient to build non-measurable sets, but is equivalent to the Baire Category Thm.

thm: A rln R on a set X is well-founded iff there does not exist an infinite decreasing chain, ie a sequence $\{x_n\}_{n \in \omega}$ st $x_{n+1} R x_n$ $\forall n \in \omega$.

Pf: If R is well-founded, and $\{x_n\}_{n \in \omega}$ is a sequence in X, then $\exists$ a "smallest" element x_n ie $\forall m \neq n$, $x_n R x_m$. In particular, $x_n R x_{n+1}$. Thus, the sequence is not strictly decreasing.

Conversely, $\not\exists$ (X, R) has no infinite decreasing chains. Then let $S \in \mathcal{P}(X) - \{\emptyset\}$. Let $x_0, \dots, x_n$ be a finite decreasing chain from S.

That is, $x_n R x_{n-1} R \cdots R x_0$. Since R has no infinite decreasing chains, $\exists m$ st $x_n R x_m$. Thus, R is entire. Let $A = \{y \in S : x_n R y\}$. By assumption $A \neq \emptyset$. By DC, there is an infinite sequence $\{x_k\}_{k \in \omega}$ st $\forall k \in \omega$, $x_n R x_k$ ie $x_k \in A$.

Something is wrong with this. $\boxed{S11}$

Def^n: If $(X, \leq_X)$ is a WO, then for $x \in X$, define $I_x := \{y \in X : y \leq x\}$ ie it's the initial segment up to x.

Def^n: An order isomorphism $\pi : X \to Y$ between two linear orderings $(X, \leq_X)$ $(Y, \leq_Y)$ is an order preserving bijection. That is, for $x, y \in X$, $x \leq_X y \Rightarrow \pi(x) \leq_Y \pi(y)$.

[This means that by assuming DC, a linear ordering is a WO iff it does not have an infinite decreasing chain, and even w/o DC, being a well order still implies there are none.]

Fact: Order Isomorphisms are Unique.

Fact: No WO is isomorphic to any initial segment of itself.

Fact: $\forall$ wo's $(X, \leq_X)$, $X \cong X$. (by the identity map.)

Fact: If $X \cong Y$ and $Y \cong Z$, then $X \cong Z$ [If $\pi : X \to Y$ and $\sigma : Y \to Z$ are the isomorphisms then $\sigma \circ \pi : X \to Z$ is an iso.]

Fact: If $\pi : X \to Y$ is an isomorphism and $\pi(x) = y$ then $\pi \restriction I_x$ is an isomorphism from I_x to I_y.

Fact: Let $(X, \leq_X)$, $(Y, \leq_Y)$ be well orderings, then either $X \cong I_y$ for some $y \in Y$, $Y \cong I_x$ for some $x \in X$, or $X \cong Y$.

Let : WO := the class of all well ordered sets. Define for $(X, \leq_X)$, $(Y, \leq_Y)$ in WO that $X \approx Y$ iff $X \cong Y$. This is reflexive and symmetric, well defined and transitive.

[Also, if we define $X \leq Y$ iff $X \cong Y$ or $\exists y \in Y$ st $X \cong I_y$, then $\leq$ is a well ordering on WO]

Informal Notion of an Ordinal: An ordinal α is an equivalence class of well orderings under the equivalence rln of order isomorphism.

That is, $\alpha = [(X, \leq_X)]$

By the fact about trichotomy, let $\alpha, \beta \in ON$. Then define $\alpha < \beta$ if for some $(X, \leq_X) \in \alpha$, some $(Y, \leq_Y) \in \beta$, $\exists y \in Y$ st $y \neq \max\{Y\}$ and $X \cong I_y$. This is well defined by our previous facts.

Def^n: A set is transitive if $\forall x \in X$ and $\forall y \in x$, $y \in X$

Ex: $\{0, \{0,1\}, 1\}$ is transitive. Removing the outer 0 or 1 makes it not transitive.

Formal Def^n: An ordinal α is a transitive set which is well ordered by the $\in$ relation.

Fact: If $\alpha \in ON$ and $\beta \in \alpha$, then $\beta = I_\beta$

Fact: If $\alpha, \beta \in ON$ and $\alpha \cong \beta$, then $\alpha = \beta$.

Fact: If $\alpha, \beta \in ON$, then either $\alpha \in \beta$, $\beta \in \alpha$, or $\alpha = \beta$.

Fact: The collection ON with the $\in$ relation is a well order. (But not a set)

Fact: Every well ordering $(X, \leq_X)$ is order-isomorphic to a unique ordinal.

We say $\alpha \in ON$ is a <u>successor ordinal</u> if $\{\beta : \beta < \alpha\}$ has a maximal elt. Otherwise, α is a <u>limit ordinal</u>.

If α is a successor ordinal, then the largest β st $\beta < \alpha$ is called the <u>predecessor</u> of α.

If $\alpha \in ON$, then let $S(\alpha) = \alpha \cup \{\alpha\}$. This is also an ordinal, (it is transitive, and wo'd by $\in$). Denote $S(\alpha) := \alpha + 1$, it is the least ordinal greater than α. it is namely the $\alpha \in S(\alpha)$, successor of α.

An ordinal α is <u>countable</u> if it is countable as a set. That is, $\exists$ a surjective function $f : \mathbb{N} \to \alpha$.

<u>Thm</u>: If $\alpha_1, \alpha_2, \ldots$ is a countable set of countable ordinals, then $\exists \beta \in ON$ st $\beta > \alpha_i$ $\forall i \in \omega$.

[This follows from a lemma saying that if $S \subseteq ON$, then $\bigcup S \in ON$, and the fact that a ctble union of ctble sets is ctble.]

<u>Def^n</u>: A limit ordinal α is said to have cofinality ω (denoted $cof(\alpha) = \omega$) if $\exists$ an increasing map $f : \omega \to \alpha$ which is unbdd (ie cofinal)

<u>Fact</u>: Every countable limit ordinal has cofinality ω.

<u>Now, onto order topologies</u>

Let $(X, \leq)$ be a linearly ordered set. Define $(a,b) := \{x \in X : a < x < b\}$.

Let $\mathcal{B}_x = \{(a,b) : a < x < b,\ a, b \in X\}$ for $x \neq \min\{X\}, x \neq \max\{X\}$.

$\mathcal{B}_x = \{[x,b) : x < b,\ b \in X\}$ for $x = \min\{X\}$

[where $[x,b) := \{b \in X : b \geq x\}$] and if $a = \max\{X\}$ then

$\mathcal{B}_x = \{(a,x] : a < x,\ a \in X\}$ where $(a,x] := \{a \in X : a \leq x\}$.

This is a filter base $\forall x \in X$:

If $x \neq \min/\max\{X\}$, then $C_1, C_2 \in \xi_x \Rightarrow C_1 = (a_1, b_1)$, $C_2 = (a_2, b_2)$
$a_1, a_2, b_1, b_2 \in X$, $a_1 < x < b_1 \wedge a_2 < x < b_2 \Leftrightarrow \max\{a_1, a_2\} < x < \min\{b_1, b_2\}$,

$$\Rightarrow C_1 \cap C_2 = \{x \in X : a_1 < x < b_1 \wedge a_2 < x < b_2\}$$
$$= \{x \in X : \max\{a_1, a_2\} < x < \min\{b_1, b_2\}\} = (\max\{a_1, a_2\}, \min\{b_1, b_2\})$$

$\in \xi_x$. Similarly, if $x = \max\{X\}$, then $C_1, C_2 \in \xi_x \Rightarrow C_1 = (a_1, x]$,
$C_2 = (a_2, x] \Rightarrow C_1 \cap C_2 = (\max\{a_1, a_2\}, x] \in \xi_x$. Case $x = \min\{X\}$
is identical. Further, the sets $C \in \xi_x$ are open: If $C = (a, b) \in \xi_x$
and $y \in (a, b)$, then $(a, b) \in \xi_y$ and $(a,b) \subseteq (a, b)$, so ✓.
If $(a, x] \in \xi_x$ and $y \in (a, x]$, then if $y < x$, $(a, x) \in \xi_y$ and $(a, x) \subseteq (a, x]$,
so $(a, x]$ is open. Finally, if $[x, b) \in \xi_x$ and $y > x$, then
$y \in (x, b) \subseteq [x, b)$ and $(x, b) \in \xi_y$, so all are open.

Ex: $(\mathbb{R}, \leq)$ with the standard ordering - The order topology on this
is just $\mathbb{R}_{std}$, since the collection of open intervals is a base
in both cases.

Def^n: Let $(X_1, \leq_{X_1}), (X_2, \leq_{X_2}), \ldots, (X_n, \leq_{X_n})$ be linear orderings.
Define the lexicagraphic order (or the dictionary order) on $\leq_\ell$
$\prod_{i=1}^{n} X_i$ by $(x_1, x_2, \ldots, x_n) \leq_\ell (y_1, y_2, \ldots, y_n)$ if $x_1 <_{X_1} y_1$ or
if $(x_1 = y_1 \wedge x_2 <_{X_2} y_2)$ or $\ldots$ or $(x_1 = y_1 \wedge x_2 = y_2 \wedge \ldots \wedge x_n \leq_{X_n} y_n)$

It can be shown that this is a linear order, and defines an
order topology on $\prod_{i=1}^{n} X_i$.

Consider $\mathbb{R}\times\mathbb{R}_{lex}$, the order topology on $\mathbb{R}_{std}\times\mathbb{R}_{std}$ under the lexicographic order. Clearly, a base for $\mathbb{R}\times\mathbb{R}_{lex}$ is $\mathcal{B}=\{(z_1,z_2): z_1, z_2\in\mathbb{R}^2: z_1 <_{lex} z_2\}$.

Note, $\mathcal{B}_2=\{\{x\}\times(c,d): x\in\mathbb{R}, c,d\in\mathbb{R}, c<d\}$ is also a base:

The sets are open since $\{x\}\times(c,d)=((x,c),(x,d))$ (the open interval from (x,c) to (x,d)), and if $U\in\tau_{lex}$, $(x,y)\in U$, then

$\exists (a_1,b_1), (a_2,b_2)\in\mathbb{R}^2$ st $(x,y)\in((a_1,b_1),(a_2,b_2))\subseteq U$

Since $((a_1,b_1),(a_2,b_2))\in\{(x,y)$ where $a_1<x<a_2$ or $a_1=x=a_2$ and $b_1<y<b_2$. In either case, $(x,y)\in((x,b_1),(x,b_2))\subseteq((a_1,b_1),(a_2,b_2))\subseteq U$,

so $\mathcal{B}_1$ is a base. This means that clearly then

$\mathcal{B}_2=\{\{x\}\times(q_1,q_2): x\in\mathbb{R}, q_1,q_2\in\mathbb{Q}, q_1<q_2\}$ is also a base.

<u>Thm</u>: $\mathbb{R}\times\mathbb{R}_{lex}$ is 1^{st} ctble, but not seperable or 2^{nd} ctble.

<u>Pf</u>: For $(x,y)\in\mathbb{R}^2$, the set $\mathcal{B}_{(x,y)}=\{B\in\mathcal{B}_2: (x,y)\in B\}$ is a nbhd base since $\mathcal{B}_2$ is a base, and countable since $\{x\}$ is fixed. Thus, we have 1^{st} ctble. $\nsubseteq$ $\mathbb{R}\times\mathbb{R}_{lex}$ were seperable. Then $\exists D=\{(x_n,y_n)\}_{n\in\omega}$ which is ctble, dense. So then $\{x_n\}_{n\in\omega}$ is also countable. Let $x\in\mathbb{R}-\{x_n\}_{n\in\omega}$. Let $U=\{x\}\times\mathbb{R}$. This set is open in $\mathbb{R}\times\mathbb{R}_{lex}$, but clearly $U\cap D=\emptyset$, $\#$. So $\mathbb{R}\times\mathbb{R}_{lex}$ is not seperable, and therefore not 2^{nd} ctble. $\boxed{//}$

Let $\alpha \in ON$. Then $(\alpha, \in)$ is a linearly ordered set, so we can define the order topology on α as before. Denote this as $[0, \alpha) = \{\beta \in ON : 0 \leq \beta < \alpha\}$. (In reality, $[0, \alpha) = \alpha$, so $[0, \alpha)$ is just notation to remind us we're thinking about a space).

Similarly, $[0, \alpha]$ is also notation for the order topology on $S(\alpha)$.

[Really, all natural numbers are ordinals, letting $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc. By the axiom of infinity, there exists an infinite ordinal. Let ω denote the least infinite ordinal. Then $\omega = \mathbb{N}$.

Fact: $\exists$ an uncountable ordinal. Let $S \subseteq ON$ be the subset of uncountable ordinals, then S has a least elt. Let ω_1 be the least uncountable ordinal. [It's only w/ CH that $|\omega_1| = |\mathbb{R}|$.]

Thus, we have the spaces $[0, \omega_1)$ and $[0, \omega_1]$.

Suppose α is a successor ordinal. Then $\forall \beta \in \alpha$, $\{\beta\}$ is open, since if $\beta = 0$, $\{0\} = [0, 1)$, and if $\beta = \max\{\alpha\}$, then $\{\beta\} = (\beta - 1, \beta]$ (where $\beta - 1$ is defined by $S(\beta - 1) = \beta$), and otherwise, $\{\beta\} = (\beta - 1, \beta + 1)$. Thus $[0, \alpha)$ is just the discrete topology on α. Similarly, $[0, \alpha]$ is just the discrete topology on $S(\alpha)$.

Thm: $[0, \omega_1)$ is 1^{st} ctble, but not 2^{nd} ctble and not separable.

Pf: 1^{st} ctble is simple, since if α is a successor ordinal, $\{\{\alpha\}\}$ is a nbhd base at α, and if α is a limit, then since $\alpha \in \omega_1$, α is ctble and thus $\mathcal{B}_\alpha = \{(\beta, \alpha] : \beta < \alpha\}$ is a ctble set

They're open since $(\beta, \alpha] = (\beta, \alpha+1)$, and if U is open with $\alpha \in U$, then $\exists\, a, b$ st $\alpha \in (a,b) \subseteq U \Rightarrow a < \alpha < b$, then $(a, \alpha] \subseteq (a,b)$, so this is a ctble nbhd base. For seperable,

$\rlap{/}S$ $D \subseteq [0, \omega_1)$ is a ctble set. Since $\forall \gamma \in D$, $\gamma < \omega_1$ ie γ is ctble, $\bigcup D$ is itself a ctble ordinal, ie $\bigcup D = I_\alpha$ for some $\alpha < \omega_1$, ie if $\beta > \alpha$, then $\beta > \gamma$ $\forall \gamma \in D$. Therefore if $U = (\alpha, \omega_1)$ which is open in $[0, \omega_1)$, then $U \cap D = \emptyset$, ie D is not dense. Since $[0, \omega_1)$ isn't seperable, it's not 2^{nd} ctble. $\boxed{\text{✓}}$

<u>Thm</u>: $[0, \omega_1]$ is not 1^{st} ctble, 2^{nd} ctble, or seperable.

The ctble bases for $\alpha \in [0, \omega_1)$ from earlier still work, so we must show that ω_1 doesn't have a ctble base. To show this, $\rlap{/}S$ $\mathcal{B} = \{B_n\}_{n \in \omega}$ is a ctble base at ω_1. Fix a B_n. Since $\omega_1 \in B_n$, and B_n is open, $\exists\, \alpha_n < \omega_1$ st $(\alpha_n, \omega_1) \subseteq B_n$. Thus we have a ctble set of ctble ordinals $\{\alpha_n\}_{n \in \omega}$. By ctble choice, $\exists\, \alpha < \omega_1$ st $\alpha_n < \alpha$ $\forall n \in \omega$. Let $U = (\alpha, \omega_1]$, which is open. By construction, $B_n \not\subseteq U$ for any B_n, but U is a nbhd of ω_1, so ⨳. Thus, $[0, \omega_1]$ is not 1^{st} ctble. Next, note that if U is open in $[0, \omega_1)$, then clearly U is open in $[0, \omega_1]$, so if D were a countable dense set in $[0, \omega_1]$, it would also be ctble, dense in $[0, \omega_1)$, which we showed cannot happen. Thus, $[0, \omega_1]$ is also not seperable and thus also not 2^{nd} ctble. $\boxed{\text{✓}}$