Dense, NWD:

A set $A \subseteq X$ in (X, τ) is dense if $\forall U \in \tau$, $U \cap A \neq \emptyset$.

A space (X, τ) is seperable if it has a countable dense set.

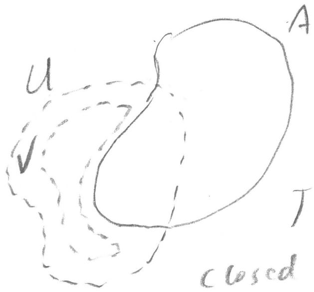
Thm: $\boxed{2^{nd} \text{ countable} \Rightarrow \text{seperable}}$

Pf: 2^{nd} ctble $\Rightarrow \exists \mathcal{B} = \{U_n\}_{n=1}^{\infty}$ which is a basis for (X, τ).
Using countable choice, pick an $x_n \in U_n$ for each $n \in \mathbb{N}$.
Then if $U \in \tau$, $\exists \{n_k\}_{k=1}^{\infty}$ st $U = \bigcup_{k=1}^{\infty} U_{n_k}$, $\{U_{n_k}\} \subseteq \mathcal{B}$. So
for any $k \in \mathbb{N}$, $U_{n_k} \subseteq U \Rightarrow x_{n_k} \in U_{n_k} \subseteq U \Rightarrow x_{n_k} \in U$,
ie $\{x_n\} \cap U \neq \emptyset$, so $\{x_n\}$ is dense in (X, τ). ∎

A set $A \subseteq X$ is nowhere dense if $\bar{A}$ contains no open sets,
ie if $\bar{A}^{\circ} = \emptyset$.

Note that $\boxed{A \text{ is nowhere dense iff } \bar{A} \text{ is nowhere dense}}$, since
$\bar{\bar{A}} = \bar{A} \Rightarrow \bar{\bar{A}}^{\circ} = \bar{A}^{\circ}$, so $\bar{\bar{A}}^{\circ} = \emptyset \Leftrightarrow \bar{A}^{\circ} = \emptyset$.

Thm: $A \subseteq X$ is nwd iff $\forall U \in \tau$ (with $U \neq \emptyset$), $\exists V \in \tau$ with $V \subseteq U$ ($V \neq \emptyset$) and $V \cap A = \emptyset$



Pf: $\boxed{\Rightarrow}$ Let $U \neq \emptyset$, $U \in \tau$. Since $\bar{A}^{\circ} = \emptyset$,
$\neg U \subseteq \bar{A} \Leftrightarrow U - \bar{A} \neq \emptyset$. Let $V = U - \bar{A}$.
This is an open set intersected with the complement of a closed set ie open. Clearly $V \subseteq U$, and $V \cap \bar{A} = \emptyset \Rightarrow V \cap A = \emptyset$.

$\boxed{\Leftarrow}$ Suppose $\exists U \in \tau - \{\emptyset\}$ st $U \subseteq \bar{A}$. (ie suppose $\bar{A}^{\circ} \neq \emptyset$). By hyp.
$\exists V \in \tau - \{\emptyset\}$ st $V \subseteq U$ and $V \cap A = \emptyset$. But $V \subseteq U \subseteq \bar{A}$,
so $V \subseteq \bar{A}$, ie $\exists x \in V \cap \bar{A}$; where V is a nbhd of x, so by /

the pt def^n of closure, $V \cap A \neq \emptyset$, ⨳. $\therefore \bar{A}^\circ = \emptyset$. $\boxed{\text{Þ}}$

<u>Thm</u>: The collection $\mathcal{I}$ of nwd sets in a space (X, τ) is an ideal. (Thus they are the natural topological notion of smallness).

<u>Pf</u>: $\emptyset$ $A \in \mathcal{I}$, and $B \subseteq A$. Then $\bar{B} \subseteq \bar{A}$, so $\bar{B}^\circ \subseteq \bar{A}^\circ = \emptyset$
$\Rightarrow \bar{B}^\circ = \emptyset$, so $B \in \mathcal{I}$. Furthermore, if $A, B \in \mathcal{I}$, then let $U \in \tau - \{\emptyset\}$, then by pvs thm, $\exists V_0 \in \tau - \{\emptyset\}$ st $V_0 \subseteq U$ and $V_0 \cap A = \emptyset$. Again by the same thm, $\exists V_1 \in \tau - \{\emptyset\}$ st $V_1 \subseteq V_0$ and $V_1 \cap B = \emptyset$. But then $(V_1 \cap A) \cup (V_1 \cap B) = V_1 \cap (A \cup B) = \emptyset$, and $V_1 \subseteq U$ where U was arbitrary, thus by the converse of the pvs thm, $A \cup B$ is nwd, so $\mathcal{I}$ is an ideal. $\boxed{\text{Þ}}$

Thus, $\mathcal{F} = \{A^c : A \in \mathcal{I}\}$ is a filter. This would be the natural topological notion of largeness. What is it?

A is nwd iff $\forall U \in \tau - \{\emptyset\}, \neg U \subseteq \bar{A}$

iff $\forall U \in \tau - \{\emptyset\}, U \cap \bar{A}^c \neq \emptyset$

$\therefore$ A is nwd iff $\bar{A}^c = (A^c)^\circ$ is dense in (X, τ).

Thus the collection of dense sets <u>whose interior is also dense</u> is the natural topological notion of largeness.

Fact: The complement of a closed, nwd set is an open dense set.

The sets whose interiors are dense are called <u>extra dense</u> (I made this up).

Fact: A set D is dense in (X, τ) iff $\bar{D} = X$.

Pf: $x \in \bar{D}$ iff $\forall U \in \tau$ st $x \in U$, $U \cap D \neq \emptyset$

Thus, if D is dense, then $\forall U \in \tau - \{\emptyset\}$, $U \cap D \neq \emptyset$, so if $x \in X$ and $U \in \tau$ st $x \in U$, $U \cap D \neq \emptyset$ ie $x \in \bar{D}$, so $X \subseteq \bar{D} \Rightarrow \bar{D} = X$ since $\bar{D} \subseteq X$. If $\bar{D} = X$ then if U is a nonempty open set, $x \in U$, then $x \in \bar{D}$ so $U \cap D \neq \emptyset$, ie D is dense. $\boxed{\text{½}}$