Various Countability Examples

Let E = the Sorgenfrey Line. It is the topology on $\mathbb{R}$ given by defining for each x the filter base $\mathfrak{F}_x = \{[x, x+\varepsilon) : \varepsilon > 0\}$

Pf this is a filter base: If $C_1, C_2 \in \mathfrak{F}_x$, then $C_1 = [x, x+\varepsilon_1)$, $C_2 = [x, x+\varepsilon_2)$ so $C_1 \cap C_2 = [x, \min\{\varepsilon_1, \varepsilon_2\}) \in \mathfrak{F}_x$. $\boxed{\varepsilon}$

Recall U is open in E iff $\forall x \in U, \exists C \in \mathfrak{F}_x$ st $C \subseteq U$.

Consider $[x, x+\varepsilon)$, let $y \in [x, x+\varepsilon)$. then if $y = x$, $[x, x+\varepsilon) \subseteq [x, x+\varepsilon)$, so ✓ otherwise, $x < y < x+\varepsilon$. $[y, y+(x+\varepsilon-y)) \in \mathfrak{F}_y$ and $[y, y+(x+\varepsilon-y)) \subseteq [x, x+\varepsilon)$

So $[x, x+\varepsilon)$ is always open. Furthermore, $[a, b)$ is always open $\forall a, b \in \mathbb{R}$. $[a,b)^c = (-\infty, a) \cup [b, \infty) = \left(\bigcup_{n=1}^{\infty} [a-n, a)\right) \cup \left(\bigcup_{n=1}^{\infty} [b, b+n)\right)$ a union of open sets, so $[a, b)$ is also closed. $\therefore$ all sets of the form $[a,b)$ are clopen.

Fact: $\mathbb{R}_{std} \subseteq E$. ie E is finer than $\mathbb{R}_{std}$.

Pf. Note that the collection $\mathcal{B}_{std} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for $\mathbb{R}_{std}$. [The sets (a, b) are always open since if $x \in (a,b)$, $a < x < b$, choose $\varepsilon = \min\{x-a, b-x\}$ so that $B_p(x, \varepsilon) \subseteq (a, b)$. $B_p(x, \varepsilon) \in \mathfrak{F}_x$ so that's the def^n of open. Furthermore, if for $x \in \mathbb{R}$, $\mathcal{B}_x = \{(a,b) \subseteq \mathbb{R} : x \in (a,b)\}$, then if $U \in \tau_{std}$, $x \in U$, $\exists B \in \mathfrak{F}_x$ ie $B = B_p(x, \varepsilon)$ st $B_p(x, \varepsilon) \subseteq U$.

But $B_p(x, \varepsilon) = \{y \in \mathbb{R} : |x-y| < \varepsilon \iff -\varepsilon < x-y < \varepsilon \iff x-\varepsilon < y < x+\varepsilon\}$

ie $B_p(x, \varepsilon) = (x-\varepsilon, x+\varepsilon)$, so let $C = (x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}) \in \mathcal{B}_x$, so $C \subseteq B_p(x, \varepsilon) \subseteq U \Rightarrow C \subseteq U$. Thus $\mathcal{B}_x$ is a nbhd base for $\mathbb{R}_{std}$, and thus $\mathcal{B}$ is a base since $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$.]

Thus, $\mathcal{B}_x^{std} = \{(a,b) : x \in (a,b)\}$ is a nbhd base for $\mathbb{R}_{std}$, and

$\mathcal{B}_x^{E} = \{[x, x+\varepsilon) : \varepsilon > 0\}$ is a nbhd base for E. If $x \in \mathbb{R}$,

$x \in (a,b)$ for some $(a,b) \in \mathcal{B}_x^{std}$, then let $\varepsilon = b - x$. Then

$x \in [x, b-x) \subseteq (a,b)$, so by the Hausdorff Criterion, $\mathbb{R}_{std} \subseteq E$.

Furthermore, $E \subsetneq \mathbb{R}_{std}$, since given $[x, x+\varepsilon) \in \mathcal{B}_x^E$, $x \in (a,b) \Rightarrow a < x$

$\Rightarrow a \notin [x, x+\varepsilon)$, so there does not exist a $B \in \mathcal{B}_x^{std}$ st $B \subseteq [x, x+\varepsilon)$.

So the sorgenfrey line is strictly finer than $\mathbb{R}_{std}$. $\boxed{\varepsilon}$

Thm: E is seperable and 1^{st} ctble, but not 2^{nd} ctble.

Pf: Consider $\mathbb{Q} \subseteq E$. If U is open, $\exists [x, x+\varepsilon) \subseteq U$.

$\exists q \in \mathbb{Q}$ st $x < q < x+\varepsilon$ [Lemma: If $a<b$, $\exists q \in \mathbb{Q}$ st $a < q < b$.

to show: $b-a > 0$ so $\exists n \in \mathbb{N}$ st $n > \frac{1}{b-a}$. Let $M = \{ l \in \mathbb{N} : \frac{l}{n} > a \}$. Since

$\mathbb{N}$ is well ordered, $\exists m = \min\{M\}$. Thus, we have that $\frac{m}{n} > a$, and

$\frac{m-1}{n} \le a$, ie $m-1 \le an$, and $m > an$. Since $n > \frac{1}{b-a}$, $\frac{1}{n} < b-a$, so

$m-1 \le an \Rightarrow m \le an+1 \Rightarrow \frac{m}{n} \le a + \frac{1}{n} < a + (b-a) = b$. $\therefore a < \frac{m}{n} < b$. $\boxed{\varepsilon}$]

So $q \in \mathbb{Q} \cap U$, ie $\mathbb{Q} \cap U \neq \emptyset$, ie $\mathbb{Q}$ is dense in E. So E is seperable.

Next, Consider $\mathcal{B}_x = \{[x, x+\frac{1}{n}) : n \in \mathbb{N}\}$. all of the $B \in \mathcal{B}_x$ are open.

Further, if U is open in E with $x \in U$, then $\exists \varepsilon > 0$ st $[x, x+\varepsilon) \subseteq U$,

and $\exists n \in \mathbb{N}$ st $\frac{1}{n} < \varepsilon \Rightarrow [x, x+\frac{1}{n}) \subseteq U$, so $\mathcal{B}_x$ is a nbhd base, ie E is

1^{st} ctble. To show E is not 2^{nd} ctble, $\mathcal{S}$ $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ were a base for

E. Note that $\forall x \in \mathbb{R}$, $[x, \infty)$ is open. $x \in [x, \infty) \Rightarrow \exists B_{n(x)}$ st

$x \in B_{n(x)} \subseteq [x, \infty)$. This means $\min\{B_{n(x)}\} = x$. If we let $n(x)$

be the least $m \in \mathbb{N}$ st this is true, then (

We've defined a 1-1 function $x \mapsto n(x)$. Thus $|\mathbb{R}| \leq \omega$, $\times$.

Thus, there does not exist a countable base. $\boxed{/\!/}$

<u>Corollary</u>: E is not metrizable

<u>Pf</u>: If E were a metric space, then E separable $\Rightarrow E$ 2^{nd} ctble, which we just showed was not the case. $\boxed{/\!/}$

We now define the <u>Scattered Line</u> by defining the following filter bases: $(\mathbb{R}, \tau_s)$ is given by $\xi_x = \{\{x\}\}$ if $x \in \mathbb{R} - \mathbb{Q}$

(The scattered line is also called the <u>Michael Line</u>) $\xi_x = \{(x-\varepsilon, x+\varepsilon) : \varepsilon > 0\}$ if $x \in \mathbb{Q}\}$.

If $x \in \mathbb{R} - \mathbb{Q}$ then ξ_x is trivially a filter base, and if $\xi_x \in \mathbb{Q}$ then we have the filter base of open balls in the std metric. Thus τ_s is a topology. Let $x \in \mathbb{R} - \mathbb{Q}$. Is $\{x\}$ open? Yes. $\{x\} \subseteq \{x\}$, and that's the only pt to check. What about $x \in \mathbb{Q}$?

Let $y \in (x-\varepsilon, x+\varepsilon)$. If $y \in \mathbb{Q}$, then of course, if $\delta = \min\{y-(x-\varepsilon), (x+\varepsilon)-y\}$ then $(y-\delta, y+\delta) \subseteq (x-\varepsilon, x+\varepsilon)$. If $y \in \mathbb{R} - \mathbb{Q}$, then $\{y\} \subseteq (x-\varepsilon, x+\varepsilon)$.

So all of our filters are nbhd bases. In fact, the same argument shows that (a,b) is open $\forall a < b \in \mathbb{R}$. So $\tau_{std} \subseteq \tau_s$.

Also note that $[a,b]$ can also sometimes be open. If $a, b \in \mathbb{R} - \mathbb{Q}$, then $[a,b] = \{a\} \cup \{b\} \cup (a,b)$, a union of open sets is open.

But $(-\infty, a) \cup (b, \infty)$ is also open, so $[a,b]$ is closed. So if a and b are irrational, (a,b) is clopen. If $a, b \in \mathbb{Q}$, then $[a,b]$ is not open, since there doesn't exist open sets U_a or U_b a or b st $U_a \subseteq [a,b]$ (or U_b). Thus $[a,b]^\circ = (a,b)$, so not open.

What about $[a, b)$ or $(a, b]$? These are open if the contained endpt is irrational. (going to stop thinking about this since it's all over the place).

<u>Thm</u>: The scattered line $\underset{\mathbb{R}_s}{\underset{\parallel}{(\mathbb{R}, \tau_s)}}$ is 1^{st} ctble, not seperable, and not 2^{nd} ctble

<u>Pf</u>: Let $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, then $\{(x-q, x+q) : q \in \mathbb{Q}\}$ is a nbhd base by the same argument used for $\mathbb{R}_{std}$.
If $x \in \mathbb{R} - \mathbb{Q}$, then $\mathfrak{F}_x = \{\{x\}\}$ is already a nbhd base. So
So $\mathbb{R}_s$ is 1^{st} ctble. To show that it isn't seperable, ρ D were a dense set. Then $D \cap \{x\} \neq \emptyset \ \forall x \in Irr \Rightarrow Irr \subseteq D$, $\#$
So not seperable. Since 2^{nd} ctble $\Rightarrow$ seperable, not seperable $\Rightarrow$ not 2^{nd} ctble. $\square$

Next we define the <u>Moore Plane</u>. Let X = the upper half plane,
ie $X = \{(a,b) \in \mathbb{R}^2 : y \geq 0\} = H$.
Then define for $z = (x, y)$ with $y > 0$ $\mathfrak{F}_z = \{B_p(z, \varepsilon) : \varepsilon > 0,\ B_p(z, \varepsilon) \subseteq H\}$
and for $z = (x, 0)$, $\mathfrak{F}_z$ is all sets of the form $\{z\} \cap D$ where D is the interior of a disk tangent to x

Clearly for $y > 0$, $\mathfrak{F}_z$ is a filter base, and in fact a nbhd base. If $z = 0$, then $C_1, C_2 \in \mathfrak{F}_z$ $\Rightarrow$ the smaller of the two will be inside the larger, so this is a filter base. So this is a valid topology.

More Rigorous Def^{n} of $\mathfrak{F}_z$:

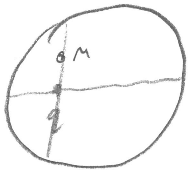
Define $U_\varepsilon(p, q) := \{(x,y) \in \mathbb{R}^2 : (x-p)^2 + (y-q)^2 < \varepsilon^2\}$ (Just the Circle centered at (p,q) of radius ε)

$V_\varepsilon(p) := \{(p, 0)\} \cup \{(x,y) : (x-p)^2 + (y-\varepsilon)^2 < \varepsilon^2\}$

Then for $Z=(x,y)$, $\xi_Z = \begin{cases} \{U_\varepsilon(x,y) : 0<\varepsilon<y\} & \text{if } y>0 \\ \{V_\varepsilon(x) : \varepsilon>0\} & \text{if } y=0 \end{cases}$

<u>Thm</u>: The Moore Plane is seperable and 1^{st} ctble but not 2^{nd} ctble.

<u>Pf</u>: For seperable, we claim $\mathbb{Q}\times\mathbb{Q}$ is dense: Let U be open in $\mathbb{R}^2_{n^+}$ (Moore Plane).
Then $\exists$ either a $U_\varepsilon(p,q)$ or a $V_\varepsilon(p)$ st $U_\varepsilon \subseteq U$ or $V_\varepsilon \subseteq U$.
In either case, there is a disc $D=\{(x,y) : (x-a)^2+(y-b)^2<r\}$ for some $a,b\in\mathbb{R}$, some $r>0$ which is a subset of U_ε or V_ε.
Then this contains the set $I_x=\{(x,y) : y=b,\ a-r<x<a+r\}$.
Choose $q\in\mathbb{Q}$ st $a-r<q<a+r$. Then for some number $s>0$,
we have that $(q,y)\in D$ $\forall\, b-s<y<b+s$. Choose $m\in\mathbb{Q}$
st $b-s<m<b+s$. Then $(q,m)\in\mathbb{Q}\times\mathbb{Q}$ and
$(q,m)\in D$ so $D\cap U\neq\emptyset$.



For 1^{st} ctble, let $Z\in\{(x,y) : y>0\}$. If $y=0$, then let
$\mathcal{B}_x=\{V_{\frac{1}{n}}(x)\}_{n=1}^{\infty}$. This is a nbhd base since the $V_{\frac{1}{n}}(x)$
are all open and $\left(V_{\frac{1}{n_1}}(x)\right)\cap\left(V_{\frac{1}{n_2}}(x)\right)=V_{\frac{1}{\min\{n_1,n_2\}}}(x)\in\mathcal{B}_x$.
$\{U_{\frac{1}{n}}(x,y)\}_{n=1}^{\infty}$ works the same if $y>0$.

For 2^{nd} ctble, $\S$ bwoc that it were. Then

<u>Lemma</u>: A subspace of a 2^{nd} ctble space is 2^{nd} ctble

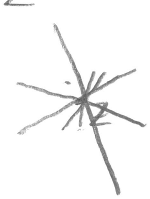
<u>Pf</u>: Let (X,τ) be a second ctble space $Y\subseteq X$ with the subspace top. Let $\mathcal{B}=\{B_n\}_{n\in\omega}$ be a ctble base for X.
Let $\mathcal{B}_Y=\{B_n\cap Y\}_{n\in\omega}$. nts this is a base for Y. $\S$ $U\subseteq Y$ is open in Y. Let $y\in U$. Then $\exists\, V\in\tau$ st $U=V\cap Y$. Then $y\in V$, so
$\exists n\in\omega$ st $y\in B_n\subseteq V \Rightarrow y\in B_n\cap Y\subseteq V\cap Y=U$. So $\mathcal{B}_Y$ is a base,
and thus Y w/ the subspace top. is 2^{nd} ctble. $\boxed{\text{Ξ}}$

Now, $\oint$ bwoc that $\mathbb{R}^2_{M^+}$ were 2^{nd} ctble, then $X = \mathbb{R}^2_{M^+} \cap \{(x,y) : y = 0\}$ is 2^{nd} ctble. Note that for any $\varepsilon > 0$, $V_\varepsilon(x)$ is open so $V_\varepsilon(x) \cap X = \{x\}$ is relatively open. Thus X as a subspace has the discrete topology. But if $\mathcal{B} = \{B_n\}_{n \in \omega}$ were a ctble base for $\mathbb{R}^2_{M^+}$, then $\mathcal{B}_X = \{B_n \cap X\}_{n \in \omega}$ would be a base for $(X, \mathcal{P}(X))$. Let $\{(a,0)\}$ be an open singleton set in X. Then $\exists n \in \omega$ st $\{(a,0)\} \subseteq (B_n \cap X) \subseteq \{(a,0)\} \Rightarrow B_n \cap X = \{(a,0)\}$. Thus if $b \neq a \in \mathbb{R}$, then the $B_m \cap X$ satisfying the same property would be $\{(b,0)\} \neq \{(a,0)\}$, so there is a unique one of these for each $x \in \mathbb{R}$, thus $|\mathcal{B}_X| > \omega$, $\#$. $\boxed{//}$

<u>Corollary</u>: $\mathbb{R}^2_{M^+}$ is not metrizable.

Now, the <u>Radial Plane</u>, $\mathbb{R}^2_{*}$ is defined as follows:

Define for $z \in \mathbb{R}^2$ a <u>star</u> about z to be a set S st for any line l through z then $l \cap S$ is a symmetrical "open interval" about z on the line l



For $z \in \mathbb{R}^2$, let $\mathcal{S}_z = \{S \subseteq \mathbb{R}^2 : S \text{ is a star about } z\}$

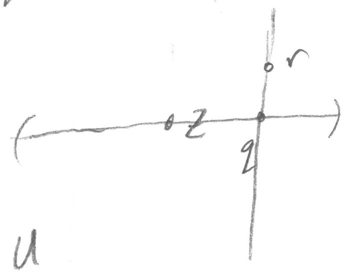
$\hookrightarrow$ More rigorously, for any $l = \{(x, ax+b) : a, b \in \mathbb{R}\}$, $z \in l$. (ie $\exists x'$ st $(x', ax'+b) = z$) $\Rightarrow \exists r > 0$ so that if $x' - r < y < x' + r$, then $(y, ay+b) \in S$.

To show this is a filter base, let S_1, S_2 be stars about z. Let l be a line through z. Then $\exists \varepsilon_1, \varepsilon_2 > 0$ st

$z = (x,y)$, $l = \{(r, ar+b) : r \in \mathbb{R}, a, b \in \mathbb{R}\}$ (ie $\exists x'$ st $(x', ax'+b) = (x,y)$)

$l \cap S_1 = \{(r, ar+b) : x' - \varepsilon_1 < r < x' + \varepsilon_1\}$ and $l \cap S_2 = \{\text{'' ''}\ \varepsilon_2\}$

Choose $q \in \mathbb{Q}$ st $(q, y) \in \ell \cap S$. Let ℓ' be the vertical line through



Assuming that $S \subseteq U$

(q, y). $(q,y) \in \ell \cap S \subseteq \ell \cap U \subseteq U$, so

There is a star T through (q, y) st

$\ell' \cap T$ is an open symmetric interval about (q, y)

with length $\varepsilon_2 > 0$. Choose $r \in \mathbb{Q}$ st $y - \varepsilon_2 < r < y + \varepsilon_2$.

Then $(q, r) \in \ell' \cap T \subseteq U$, so $(q,r) \in \mathbb{Q} \times \mathbb{Q} \cap U$, ie

$\mathbb{Q} \times \mathbb{Q} \cap U$ is nonempty. So $\mathbb{Q} \times \mathbb{Q}$ is dense, ie $\mathbb{R}^2_*$ is seperable.

Next, we need the following lemma:

<u>Lemma</u>: If $S \subseteq \mathbb{R}^2$ and no 3 pts in S are colinear, then S is closed in $\mathbb{R}^2_*$.

<u>Pf</u>:

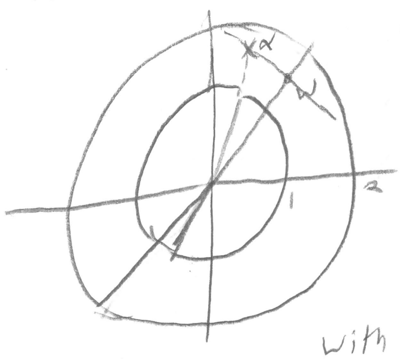
Thus letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ will clearly work, so $S_1 \cap S_2 \in \mathcal{S}_z$.

Therefore this works as a filter base.

Are stars themselves open in $\mathbb{R}^2_*$? Let S be a star about $z = (x, y)$.

Specifically consider the star S about $(0,0)$ defined by:

If ℓ is the line about θ, then let $\ell \cap S$ be the interval of length 1 if θ is irrational, and 2 if θ is rational. We claim S is not open. Indeed, if w is any point in the annulus centered at O w/ inner radius 1 and outer radius 2 and also in S, then there is no way to define a star $S_w \in \mathcal{S}_w$ st $S_w \subseteq S$. If there were such a star, Let θ' be any irrational angle. Then the line through w at angle θ' would have to intersect S_w with some symmetric interval about w.



Choose $\alpha = (r, \theta)$ st θ is irrational, and α is on this symmetric interval. Then the line through O at angle θ only intersects S up to $r = 1$, and if this α is still in the annulus (which we can assume WLOG) then this line cannot hit α! So $S_w \not\subseteq S$, for any star S_w, and thus S is not open.

However, If we define Stars by letting all intersection lengths be the same, then we just have open balls, so all open balls are stars, and are open, (by the same argument as usual) so $\mathbb{R}^2_{std}$ is coarser than $\mathbb{R}^2_*$.

<u>Thm</u>: $\mathbb{R}^2_*$ is Seperable, but not 1^{st} ctble and not 2^{nd} ctble.

<u>Pf</u>: <u>Seperable</u>: We claim $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2_*$.

Let U be open in $\mathbb{R}^2_*$. Let $z = (x,y) \in U$. Let ℓ be the horizontal line through z. Let $\varepsilon > 0$ be length of the symmetric open interval $\ell \cap S$.