More Basic Notions

If τ_1 and τ_2 are topologies on X, then τ_2 is <u>finer</u> than τ_1 if $\tau_1 \subseteq \tau_2$, (and τ_1 is <u>courser</u> than τ_2)

2 definitions of being a <u>nbhd</u> of $x \in X$

<u>Standard</u>: A is a nbhd of x if $x \in A \in \tau$.

<u>Nonstandard</u>: A is a nbhd of x if $x \in A^\circ$

Under the nonstandard def^n, the nbhds of x form a filter called the <u>nbhd filter</u> $\mathcal{U}_x$ (also known as the <u>nbhd system</u>

<u>Def^n</u>: If (X, τ) a top. space, $x \in X$, then $\mathcal{B}_x$ is a <u>nbhd base</u> at x if it is a collection of open nbhds of x st if U is a (std) nbhd of x, then $\exists B \in \mathcal{B}_x$ st $B \subseteq U$. [ie $\mathcal{B}_x$ is a filter base of open sets for the nbhd filter] <u>Note</u>: Willard defines this w/o the restriction of being open.

A space is <u>1^{st} countable</u> if $\forall x \in X$, $\exists$ a countable nbhd base $\mathcal{B}_x$.

this shows this

<u>Facts about nbhd bases</u>: 1) $\boxed{\text{If } V_1, V_2 \in \mathcal{B}_x, \exists V_3 \in \mathcal{B}_x \text{ st } V_3 \subseteq V_1 \cap V_2}$

<u>Pf</u>: $\mathcal{U}_x = \{A \subseteq X : \exists B \in \mathcal{B}_x \text{ st } B \subseteq A\}$. So $V_1, V_2 \in \mathcal{B}_x \Rightarrow V_1, V_2 \in \mathcal{U}_x$ and since $\mathcal{U}_x$ is a filter, $V_1 \cap V_2 \in \mathcal{U}_x \Rightarrow \exists V_3 \in \mathcal{B}_x$ st $V_3 \subseteq V_1 \cap V_2$. ▯

$\boxed{\text{2) } U \in \tau \text{ iff } U \text{ contains a basic nbhd of each of it's points.}}$

That is, $\forall x \in U$, $\exists V \in \mathcal{B}_x$ st $V \subseteq U$

<u>Pf</u>: $U \in \tau \Rightarrow U \in \mathcal{U}_x$ $\forall x \in U$, so $\exists B \in \mathcal{B}_x$ with $B \subseteq U$

Conversely, if $\forall x \in U$, $\exists V_x \in \mathcal{B}_x$ st $V_x \subseteq U$, then $U = \bigcup_{x \in U} V_x$, a union of open sets, ie open. ▯

[Note that 1) also proves that $\mathcal{U}_x$ is indeed a filter.]

Recall that in general, $\xi \subseteq \mathcal{P}(\mathbb{X}) - \{\emptyset\}$ is a base for some filter $\mathcal{F}$ iff $\forall C_1, C_2 \in \xi$, $\exists C_3 \in \xi$ st $C_3 \subseteq C_1 \cap C_2$.

<u>Thm (General Way to define a topology)</u>: Suppose for each $x \in \mathbb{X}$ that $\xi_x \subseteq \mathcal{P}(\mathbb{X}) - \{\emptyset\}$ is a filter base st $\forall C \in \xi_x$, $x \in C$.

Define $U \in \tau \iff \forall x \in U, \exists C \in \xi_x$ st $C \subseteq U$. Then τ is a topology on $\mathbb{X}$. [Note that the $C \in \xi_x$ are not necessarily open! Thus, ξ_x is not necessarily a nbhd base at x!]

However, if $(\mathbb{X}, \tau)$, then the 'natural' ξ_x to define is $\xi_x = \{U \in \tau : x \in U\}$. This of course is a nbhd base.

<u>Pf</u>: $\emptyset \in \tau$ vacuously, and $\mathbb{X} \in \tau$ is trivial. Let $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$. Let $U = \bigcup_{\alpha \in I} U_\alpha$. Let $x \in U$. Then $\exists \alpha \in I$ st $x \in U_\alpha$. Thus $\exists C \in \xi_x$ st $C \subseteq U_\alpha$. But then $C \subseteq \bigcup_{\alpha \in I} U_\alpha = U$, so $U \in \tau$.

Finally, if $U, V \in \tau$, $x \in U \cap V$, then $x \in U \Rightarrow \exists C_1 \in \xi_x$ st $C_1 \subseteq U$, $x \in V \Rightarrow \exists C_2 \in \xi_x$ st $C_2 \subseteq V$, but since ξ_x is a filter base, $\exists C_3 \in \xi_x$ st $C_3 \subseteq C_1 \cap C_2 \subseteq U \cap V \Rightarrow U \cap V \in \tau$. $\boxtimes$

We can define $\overline{A}$ and A° in terms nbhd bases:

If $\forall x \in \mathbb{X}$, $\mathcal{B}_x$ is a nbhd base for $(\mathbb{X}, \tau)$, then

$$E^\circ = \{x \in \mathbb{X} : \exists B \in \mathcal{B}_x \text{ st } B \subseteq E\}$$

$$\overline{E} = \{x \in \mathbb{X} : \forall B \in \mathcal{B}_x,\ B \cap E \neq \emptyset\}.$$

Let A be a set in $(X, \mathcal{T})$. $x \in A$ is a cluster pt or an accumulation pt or a limit pt if $\forall$ nbhds B of x, $\exists y \in A$ st $y \neq x$. ie $A \cap B \not\subseteq \{x\}$.

The set of limit pts is denoted A', it is the derived set.

Note: $x \in A \not\Rightarrow x \in A'$! Ex $B = \{\frac{1}{n} : n \in \omega\} \subseteq \mathbb{R}_{std}$. all pts are isolated (that is, not in A'). $A' = \{0\}$.

Thm: $\bar{A} = A \cup A'$

Pf: $x \in A' \Rightarrow$ every nbhd of x intersects A, so by the basic nbhd defn of $\bar{A}$, $A' \subseteq \bar{A}$. Thus, $A \cup A' \subseteq \bar{A}$. On the other hand, if every basic nbhd U of x meets A (ie $x \in \bar{A}$) then either $x \in A$ or $\exists y \in A$ st $y \neq x$, $y \in A \cap U$, ie $x \in A'$. So $\bar{A} \subseteq A \cup A'$.

Thus $\bar{A} = A \cup A'$. $\boxtimes$

If $(X, \mathcal{T})$ is a space, then a basis for $\mathcal{T}$ is a collection $\mathcal{B}$ st $\mathcal{T} = \{\bigcup_{B \in \mathcal{C}} B : \mathcal{C} \subseteq \mathcal{B}\}$. Ie if every open set in $\mathcal{T}$ is a union of subcollections in $\mathcal{B}$. (and every union of a subcollection is in $\mathcal{T}$)

Thm: $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ iff

1) $X = \bigcup_{B \in \mathcal{B}} B$ and 2) If $B_1, B_2 \in \mathcal{B}$, and $p \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ st $p \in B_3 \subseteq B_1 \cap B_2$.

Pf: $\boxed{\Rightarrow}$ If $\mathcal{B}$ is a basis for $\mathcal{T}$, then 1) is trivial and 2) is true since $B \in \mathcal{B} \Rightarrow B \in \mathcal{T}$, so $B_1 \cap B_2 \in \mathcal{T}$, ie $B_1 \cap B_2 = \bigcup_{B \in \mathcal{C}} B$ for some $\mathcal{C} \subseteq \mathcal{B}$. If $p \in B_1 \cap B_2$, then $p \in B'$ for some particular $B' \in \mathcal{C}$, so $p \in B' \subseteq B_1 \cap B_2$.

$\boxed{\Leftarrow}$) Let $\mathcal{T}$ be the definition, wts it's a topology.

Let $\{U_\alpha\}_{\alpha\in I} \subseteq \mathcal{T}$. Each $U_\alpha = \bigcup_{\beta\in\mathcal{D}_\alpha} B_\beta$ for some $\mathcal{D}_\alpha \subseteq \mathcal{B}$. So

$\bigcup_{\alpha\in I} U_\alpha = \bigcup_{\alpha\in I}\bigcup_{\beta\in\mathcal{D}_\alpha} B_\beta \in \mathcal{T}$. Next, let $U, V \in \mathcal{T}$. Then

$U = \bigcup_{B\in\mathcal{B}_1} B$, $V = \bigcup_{C\in\mathcal{B}_2} C$, so $\left(\bigcup_{B\in\mathcal{B}_1} B\right)\cap\left(\bigcup_{C\in\mathcal{B}_2} C\right) = \bigcup_{B\in\mathcal{B}_1}\bigcup_{C\in\mathcal{B}_2}(B\cap C)$.

But by 2), for each $p\in B\cap C$, $\exists D_p \in \mathcal{B}$ st $p\in D_p \subseteq B\cap C$.

Thus $B\cap C = \bigcup_{p\in B\cap C} D_p$, so this is a big union of basis sets, ie $\in \mathcal{T}$.

$\emptyset = \bigcup_{B\in\emptyset} B$, and $X = \bigcup_{B\in\mathcal{B}} B$ by condition 1), so $\mathcal{T}$ is a topology. $\boxtimes$

- A basis can be essentially seen as what one gets by reducing the definition of a topology to only have the finite intersection property.

<u>Thm</u>: Let $\mathcal{B}$ be a collection of open sets in X. Then $\mathcal{B}$ is a basis for $(X,\mathcal{T})$ iff $\forall x\in X$, the collection $\mathcal{B}_x = \{B\in\mathcal{B} : x\in B\}$ is a nbhd base at x.

<u>Pf</u>: $\boxed{\Rightarrow}$) For $x\in X$, def $\mathcal{B}_x = \{B\in\mathcal{B} : x\in B\}$. Clearly, each $B\in\mathcal{B}$ is a nbhd of X since by hyp. their all open and $B\subseteq \mathcal{B}$.

If U is a nbhd of x, then $x\in U^\circ$. U° is open, so since $\mathcal{B}$ a basis, $U^\circ = \bigcup_{B\in\mathcal{C}} B$ for some $\mathcal{C}\subseteq\mathcal{B}$. So $\exists$ a particular $B'\in\mathcal{C}$

$x\in B' \subseteq U^\circ \subseteq U$. Thus $\mathcal{B}_x$ is a filter base for the nbhd filter, ie a nbhd base at x.

$\boxed{\Leftarrow}$ If $\mathcal{B}_x$ is an open nbhd base at x, and $\mathcal{B} := \bigcup_{x \in X} \mathcal{B}_x$, then if U is open, $p \in U$, $\exists B_p \in \mathcal{B}$ st $p \in B_p \subseteq U$. So $U = \bigcup\{B_p : p \in U\}$

This is a union of elts of $\mathcal{B}$, so $\mathcal{B}$ is a basis for $(X, \mathcal{T})$. $\boxed{\equiv}$

A space is <u>2^{nd} countable</u> if it has a countable basis.

<u>Thm</u>: 2^{nd} ctble $\Rightarrow$ 1^{st} ctble

<u>Pf</u>: 2^{nd} ctble $\Rightarrow \exists \mathcal{B} = \{U_n\}_{n \in \mathbb{N}}$. If $x \in X$, then by the above thm, $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a nbhd base, and

$\mathcal{B}_x \subseteq \mathcal{B} \Rightarrow |\mathcal{B}_x| \leq |\mathcal{B}| = \omega$. $\boxed{\equiv}$

<u>Def^n</u>: A <u>subbasis</u> for $\mathcal{T}$ is a collection $\mathcal{C} \subseteq \mathcal{T}$ st the collection

$\mathcal{B} = \{\bigcap_{i=1}^{n} B_i : n \in \mathbb{N}, B_i \in \mathcal{C}\}$ is a basis for $\mathcal{T}$

(ie the collection of all finite intersections of sets in $\mathcal{C}$ forms a basis)

<u>Thm</u>: Any nonempty collection of subsets in X is a subbase for some topology $\mathcal{T}$ on X

<u>Pf</u>: Let $\mathcal{C} \subseteq \mathcal{P}(X)$. $\emptyset \in \mathcal{T}$? Yes, since $\emptyset \in \mathcal{B}$ since it's the finite intersection of no sets, ie $\bigcap_{B \in \emptyset} B$ where $\emptyset = \emptyset$. $X \in \mathcal{T}$?

Yes, since