Topology Notes Redux

$(\mathfrak{X}, \tau)$ a top $\Leftrightarrow$ $\tau \subseteq \mathcal{P}(\mathfrak{X})$

(a) $\emptyset, \mathfrak{X} \in \tau$ (b) $\{U_\alpha\}_{\alpha \in I} \subseteq \tau \Rightarrow \bigcup U_\alpha \in \tau$

(c) $U_1, \ldots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{n} U_i \in \tau$.

These axioms characterize a topology by it's <u>open sets</u>

Alternatively, if τ are the open sets in a space, let λ be the <u>closed sets</u>, $\lambda = \{U^c : U \in \tau\}$. Axioms for closed sets are

(A) $\emptyset, \mathfrak{X} \in \lambda$ (B) $\{F_\alpha\}_{\alpha \in I} \subseteq \lambda \Rightarrow \bigcap_{\alpha \in I} F_\alpha \in \tau$.

(C) $F_1, \ldots, F_n \in \lambda \Rightarrow \bigcup_{i=1}^{n} F_i \in \lambda$.

Then (1) $\Leftrightarrow$ (i) since $\emptyset, \mathfrak{X} \in \lambda \Leftrightarrow \emptyset^c, \mathfrak{X}^c \in \tau$
$\Leftrightarrow \emptyset, \mathfrak{X} \in \tau$.

(2) $\Leftrightarrow$ (ii) since $\{U_\alpha\}_{\alpha \in I} \subseteq \tau \Leftrightarrow \{U_\alpha^c\}_{\alpha \in I} \subseteq \lambda$,

and $\bigcup_{\alpha \in I} U_\alpha \in \tau \Leftrightarrow \left(\bigcup_{\alpha \in I} U_\alpha\right)^c \in \lambda \Leftrightarrow \bigcap_{\alpha \in I} U_\alpha^c \in \lambda$.

(3) $\Leftrightarrow$ (iii) in the same way.

<u>Closure and Interior</u>:

$$A^\circ = \text{int}(A) := \bigcup\{U \in \tau : U \subseteq A\} \quad (A \subseteq \mathfrak{X})$$
$$\bar{A} = cl(A) := \bigcap\{F \in \lambda : A \subseteq F\}.$$

<u>Properties of Closure</u>:

1) $A \subseteq \bar{A}$
2) $\bar{\bar{A}} = \bar{A}$
3) $\overline{A \cup B} = \bar{A} \cup \bar{B}$
4) $\bar{\emptyset} = \emptyset$

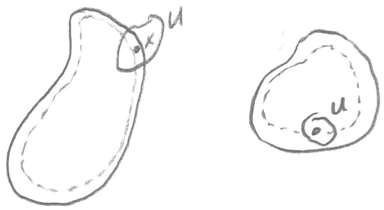
<u>Properties of Interior</u>:

i) $A^\circ \subseteq A$
ii) $(A^\circ)^\circ = A^\circ$
(iii) $(A \cap B)^\circ = A^\circ \cap B^\circ$
(iv) $\mathfrak{X}^\circ = \mathfrak{X}$

Other properties

Pt definitions:

- $x \in \bar{A}$ iff $\forall U \in \tau$ with $x \in U$, $U \cap A \neq \emptyset$
- $x \in A^\circ$ iff $\exists U \in \tau$ st $x \in U$, $U \subseteq A$



Closure characterizes closed sets: A closed iff $\bar{A} = A$

Interior characterizes Open sets: A open iff $A^\circ = A$.

Pf: • $A^\circ \subseteq A$, and A° is always open, so $A^\circ = A \Rightarrow A$ is open.

If A is open, then $A^\circ = A$ clearly.

• $\bar{A}$ is always closed, so if $\bar{A} = A$, then A is closed.

If A is closed, then clearly $\bar{A} = A$.

Relationship: $(\bar{A})^c = (A^c)^\circ$ and $(A^\circ)^c = \overline{(A^c)}$.

Topology is Characterized by a Notion of Closure: Let $p : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function satisfying the 4 closure properties. Then $\exists$ a unique topology (X, τ) st $\forall A \subseteq X$, $\bar{A} = p(A)$.

Pf: Define τ by $U \in \tau$ iff $p(X - U) = X - U$. Clearly this is uniquely defined by τ. We show τ is a topology:

$p(\emptyset) = \emptyset \Rightarrow X - \emptyset = X \in \tau$. Since $X \subseteq p(X)$, $p(X) = X$
$\Rightarrow X - X = \emptyset \in \tau$, so $\emptyset$ and X are in τ. ✓

Next, let $U, V \in \tau$. Then $p(X - U) = X - U$, $p(X - V) = X - V$.

so $p(X - (U \cap V)) = p((X \cap U^c) \cup (X \cap V^c)) = p(X - U) \cup p(X - V)$
$= (X - U) \cup (X - V) = X - (U \cap V)$, so $U \cap V \in \tau$.

Inductively this yields $U_1, \dots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{n} U_i \in \tau$

Finally, $\not\exists$ $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$. Then $P(X - \bigcup_{\alpha \in I} U_\alpha) = P(\bigcap_{\alpha \in I}(X - U_\alpha))$.

Note $\bigcap_{\alpha \in I}(X - U_\alpha) \subseteq X - U_\alpha$ for any $\alpha \in I$.

Also, in general, if $A \subseteq B \subseteq X$, $B = A \cup (B - A)$

$$\Rightarrow P(B) = P(A \cup (B-A))$$

$$= P(A) \cup P(B-A) \supseteq P(A)$$

ie $P(A) \subseteq P(B)$

so $P(\bigcap_{\alpha \in I}(X - U_\alpha)) \subseteq P(X - U_\alpha)$ for any $\alpha \in I$.

$= X - U_\alpha$ since each $U_\alpha \in \tau$.

$\therefore$ $P(X - \bigcup_{\alpha \in I} U_\alpha) = P(\bigcap_{\alpha \in I}(X - U_\alpha)) \subseteq \bigcap_{\alpha \in I}(X - U_\alpha) = X - \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau$. So τ is a Topology. $\square$

<u>Topology is Characterized by a Notion of Interior</u>: Let $Q: P(X) \to P(X)$ be a function satisfying the 4 interior properties. Then $\exists$ a unique topology (X, τ) st $\forall A \subseteq X$, $A^\circ = Q(A)$.

<u>Pf</u>: Define $P(A) := (Q(A^c))^c$. This is a function which satisfies the closure properties: $Q(A) \subseteq A \Rightarrow Q(A^c) \subseteq A^c \Rightarrow (Q(A^c))^c \supseteq A$

$\Rightarrow P(A) \subseteq A$. $Q(Q(A)) = A \Rightarrow P(P(A)) = P((Q(A^c))^c)$

$$= (Q(((Q(A^c))^c)^c))^c = (Q(Q(A^c)))^c = (A^c)^c = A$$

$Q(A \cap B) = Q(A) \cap Q(B) \Rightarrow P(A \cup B) = (Q((A \cup B)^c))^c = (Q(A^c \cap B^c))^c$

$$= (Q(A^c) \cap Q(B^c))^c = (Q(A^c))^c \cup (Q(B^c))^c$$

$$= P(A) \cup P(B).$$

Finally, $Q(X) = X \Rightarrow P(\emptyset) = (Q(X))^c = X^c = \emptyset$.

So there is a unique topology $\mathcal{T}$ for which $P(A) = \bar{A}$.

But then $P(A^c) = (Q(A))^c = \bar{A}^c = (A^\circ)^c$

$$\Rightarrow (Q(A))^c = (A^\circ)^c \Rightarrow Q(A) = A^\circ \text{ for any } A \subseteq X. \quad \square$$

Thus, a topology can be fully characterized by having any of the following.

- A collection of "open" sets satisfying (a), (b), and (c).
- A collection of "closed" sets satisfying (A), (♡), and ([illegible])
- A "closure" function on $\mathcal{P}(X)$ satisfying (1), (2), (3) and (4)
- An "interior" function on $\mathcal{P}(X)$ satisfying (i), (ii) (iii), and (iv)

Ie having any one of these defines a topology which uniquely specifies the 3 others.

Pf of Closure Properties:

3) ~~Note that $A \subseteq A \cup B \subseteq \overline{A \cup B}$~~ (~~since $cl(A \cup B)$ is the intersection of all sets containing $A \cup B$~~)

Let F be a closed set containing $A \cup B$.

Then $A \subseteq F$. Thus, $\{F \in \mathcal{C} : A \cup B \subseteq F\} \subseteq \{F \in \mathcal{C} : A \subseteq F\}$

ie $\bigcap\{F \in \mathcal{C} : A \subseteq F\} \subseteq \bigcap\{F \in \mathcal{C} : A \cup B \subseteq F\}$

ie $\bar{A} \subseteq \overline{A \cup B}$. Similarly, $\bar{B} \subseteq \overline{A \cup B}$

so $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. Next note $\bar{A} \cup \bar{B}$ is closed, with

$A \subseteq \bar{A}$, $B \subseteq \bar{B}$, so $A \cup B \subseteq \bar{A} \cup \bar{B}$. Thus

$\overline{A \cup B} = \bigcap\{F \in \mathcal{C} : A \cup B \subseteq F\} \subseteq \{$any single closed F st $A \cup B \subseteq F\}$ such as $\bar{A} \cup \bar{B}$.

So $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$, and thus $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

4) Since $\emptyset$ is itself closed, $\emptyset = \bar{\emptyset}$.

1) is trivial. 2) Clearly, $\bar{A} \subseteq \bar{\bar{A}}$. Let $x \in \bar{\bar{A}}$. By the pt definition of closure (proven below), let $U \in \mathcal{T}$ st $x \in U$.

Then $U \cap \bar{A} \neq \emptyset$, ie $\exists y \in U$ st $y \in \bar{A}$. But then U is open and contains $y \in \bar{A}$, so again by the pt definition $U \cap A \neq \emptyset$. $\therefore$ if $x \in \bar{\bar{A}}$, $U \in \mathcal{T}$ st $x \in U$, then $U \cap A \neq \emptyset$

ie $x \in \bar{A}$. So $\bar{\bar{A}} \subseteq \bar{A}$, ie $\bar{A} = \bar{\bar{A}}$.

Pf of pt definition of Closure: Suppose $x \notin \bar{A}$. Then $U = X - \bar{A}$ is open, $x \in U$, and $U \cap A = (X - \bar{A}) \cap A = X \bar{A}^c A = \bar{A}^c A = \emptyset$.

Conversely, if $\exists U \in \mathcal{T}$ st $x \in U$, $U \cap A = \emptyset$, then $X - U$ is closed, with $A \subseteq U$ and $x \notin X - U$ so $x \notin \bar{A}$. Thus

... we have $x \notin \bar{A}$ iff $\exists U \in \tau$ st $x \in U$, $U \cap A = \emptyset$

$\iff$ $x \in \bar{A}$ iff $\forall U \in \tau$ with $x \in U$, $U \cap A \neq \emptyset$

Pf of Interior Properties: $\cancel{\overline{A \cup B} = \bar{A} \cup \bar{B}}$.

$\cancel{(A \cap B)^\circ =}$ So $\cancel{((A \cap B)^c)^\circ = (\overline{A \cup B})^c}$

$\cancel{(\overline{A \cap B})^\circ = (\overline{A \cup B})^c = \bar{A}^c \cap \bar{B}^c}$

$\cancel{= \bar{A} \cap \bar{B}}$

(iii)

Since $(\bar{A})^c = (A^c)^\circ$ and $(A^\circ)^c = \overline{(A^c)}$, we have $(A \cap B^\circ)^c = \overline{((A \cap B)^c)}$

$= \overline{A^c \cup B^c} = \overline{A^c} \cup \overline{B^c} = (A^\circ)^c \cup (B^\circ)^c$

$\therefore$ $(((A \cap B)^\circ)^c)^c = ((A^\circ)^c \cup (B^\circ)^c)^c$

iff $(A \cap B)^\circ = ((A^\circ)^c)^c \cap ((B^\circ)^c)^c = A^\circ \cap B^\circ$.

(i) is trivial. (iv) is true since $X \in \tau \iff X^\circ = X$.

(ii) $((A^\circ)^\circ)^c = \overline{(A^\circ)^c} = \overline{(\overline{A^c})} = \overline{\overline{A^c}} = \overline{A^c} = (A^\circ)^c$

$\therefore$ $(A^\circ)^\circ = A^\circ$.

Pf that $(\bar{A})^c = (A^c)^\circ$: By the pt definitions of closure and intersection,

$x \in (\bar{A})^c$ iff $\neg(\forall U \in \tau$ with $x \in U$, $U \cap A \neq \emptyset)$

iff $\exists U \in \tau$ with $x \in U$ st $U \cap A = \emptyset$

iff $\exists U \in \tau$ with $x \in U$ st $U \subseteq A^c$ iff $x \in (A^c)^\circ$.

$U \cap A^c = \emptyset$

Pf that $(A^\circ)^c = \overline{(A^c)}$: $x \in (A^\circ)^c$ iff $\neg(\exists U \in \tau$ with $x \in U$, st $U \subseteq A)$

iff $\forall U \in \tau$ with $x \in U$, st $U \cap A^c \neq \emptyset$

iff $x \in \overline{(A^c)}$.

Pf of pt defⁿ of interior: Trivial. $x \in A^\circ$ iff $x \in \bigcup\{U \in \tau : U \subseteq A\}$

iff $\exists U \in \tau$ w/ $U \subseteq A$ st $x \in U$.