1 Graphing Complex Functions

Complex numbers are two dimensional. In fact, it is perfectly valid to think of them as simply vectors in $\mathbb{R}^2$. The functions that we are used to dealing with are functions of real numbers. That is, they take a single real number as an input, and return a single real number as an output:

$$f : \mathbb{R} \to \mathbb{R}$$

This lends itself well to a handy two dimensional *visualization* of a function which we are very used to - We view every number on the x-axis as a potential input, and on a separate y-axis, we plot the value of $f(x)$. The end result looks something like this:

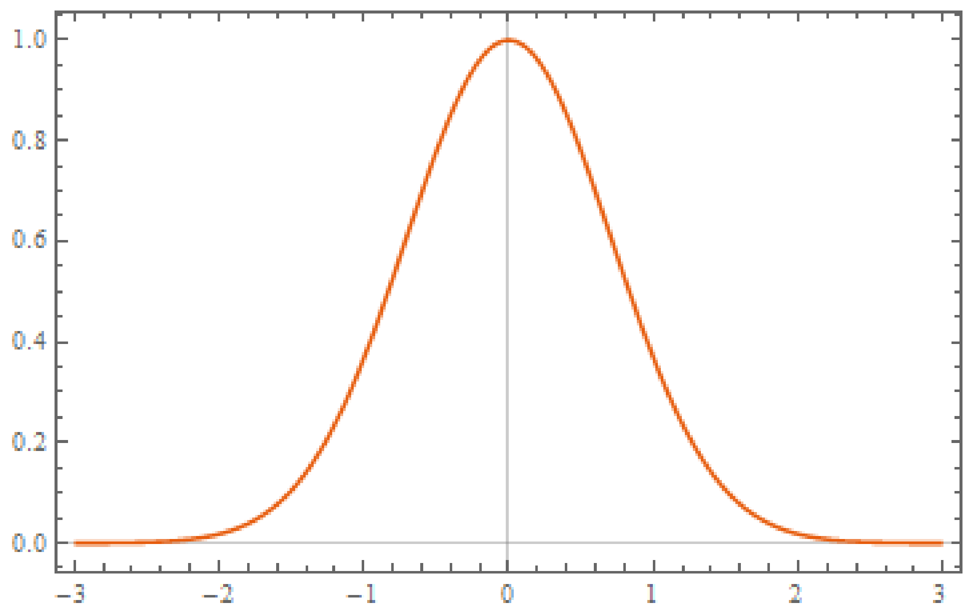


Figure 1: Graph of $f(x) = e^{-x^2}$

On the other hand, complex valued functions are going to take two real numbers as an input, an return two real numbers as an output:

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

These things are four dimensional! How could we possibly visualize them? There are lots of different ways, all of which have their uses here and there.

But one that I find particularly cool is called a **domain coloring**. The fastest way to explain it would probably be to just dive into an example. Consider the complex function $f(z) = e^z$. It's going to be advantageous for us to think of this function as taking an input in rectangular coordinates, and returning an output in polar. That is:

$$f(x + iy) = re^{i\theta}$$

Note that for a complex number $z = x + iy$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$\implies \text{r=e}^x \text{ and } \theta = y$$

The exponential function is surprisingly easy to think about as a complex function. It simply exponentiates the real part and takes that as output's magnitude, and takes the imaginary part as the angle, leaving it untouched. Let's have a look at it's domain coloring and try to figure out what's going on:

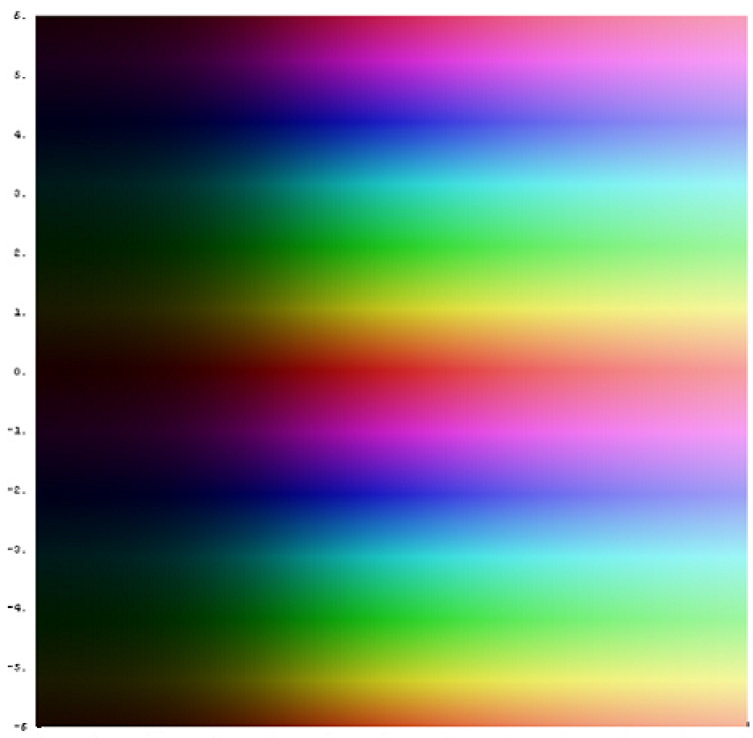


Figure 2: Domain coloring of $f(z) = e^z$

Here's how to understand this: Put your finger somewhere on the graph. You're pointing at a specific complex number - that's your input to the function. The output of the function is represented by the *color and brightness of the pixel at that point*. Darker colors correlate with magnitudes (that is, values of r) which are close to zero, and brighter colors correlate with magnitudes which are very large. We can see this in our graph - moving right corresponds with larger real inputs, which correspond to more brightness. The actual color specifies angle, according to the standard color wheel:

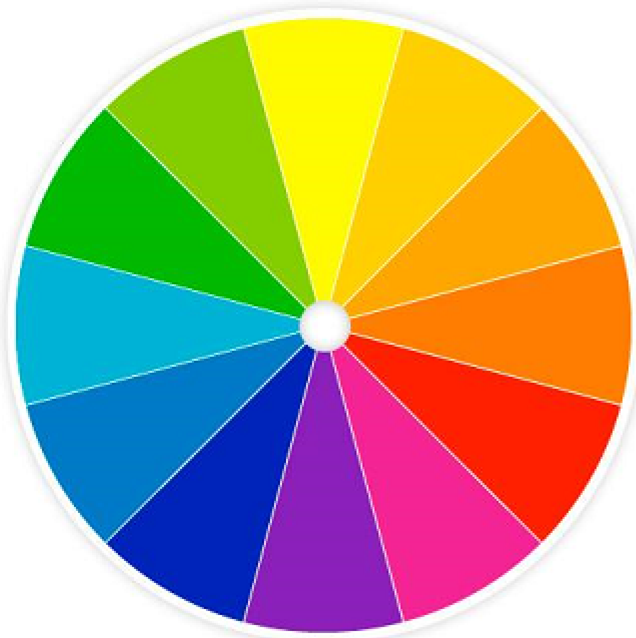


Figure 3: Standard(?) Color wheel

So reddish pixels correspond to angles which are closer to 0, yellowish pixels correspond to angles which are closer to $\frac{pi}{2}$, and so forth. Note how moving up and down on our graph, that is, shifting the imaginary part of our input up and down, we're seeing rotation around the color wheel. Neat! Let's next take a look at $sin(z)$, which looks a little more complicated:

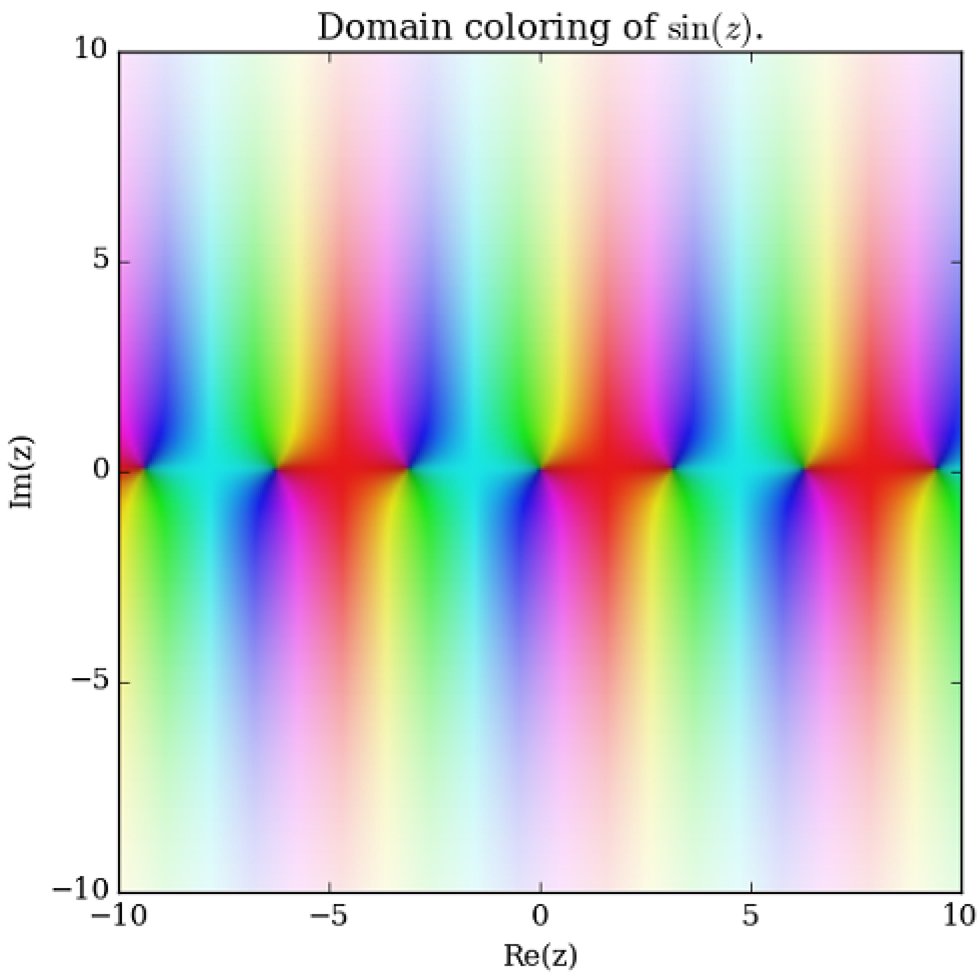


Figure 4: Yup

If you read through the hyperbolic trig packet, this image can make a lot of sense. Note that positive real numbers are indicated by the color red, and negative reals are indicated by a light blue. If you look at the horizontal line through the middle at $Im(z) = 0$, you'll can see a pattern of red-blue-red-blue indicative of the standard oscillation that you're used to. But then things get really interesting if we look vertically. Do you see the the graphs of both $sinh(x)$ and $cosh(x)$ hidden here? Try explaining some of it to yourself!

Recall that the **gamma function** is defined as the following:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{1}$$

This guy is a generalization of the factorial function to not just all real numbers greater than 0, but in fact all complex numbers except for the non-positive integers! This is true in the sense that for any positive integer n, $\Gamma(n) = (n-1)!$. It's graph for real numbers looks like this:

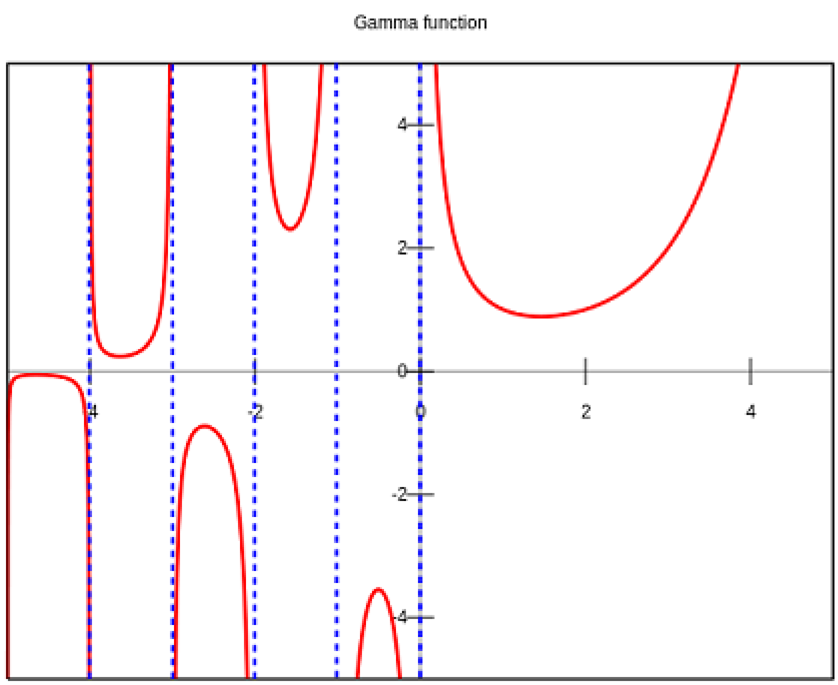


Figure 5: Strangely, the integral only fails to converge at negative integers.

This is okay I guess, but pretty colors are more my style. Here's the domain coloring:

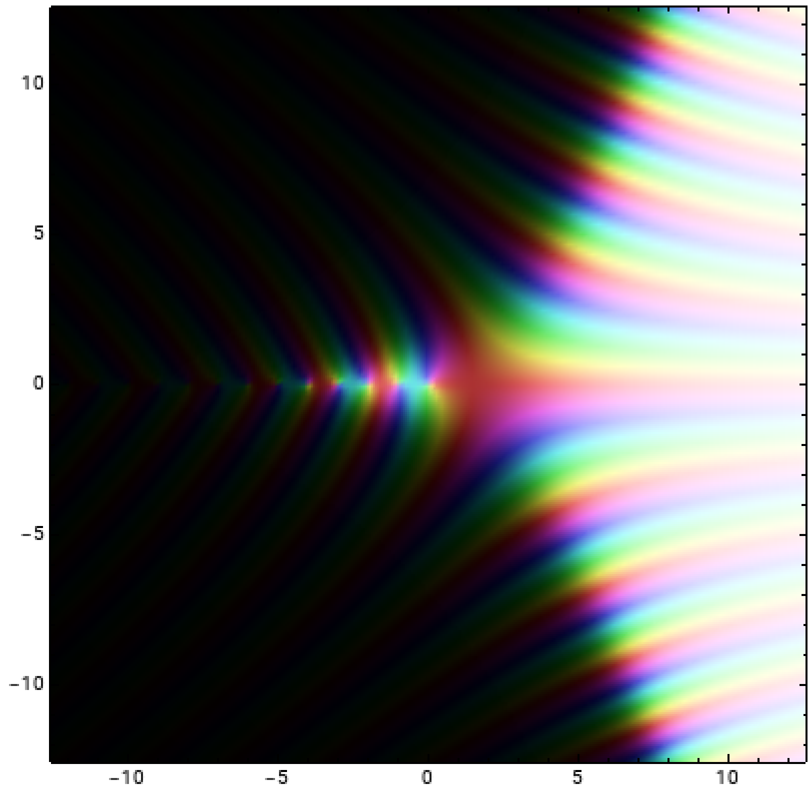
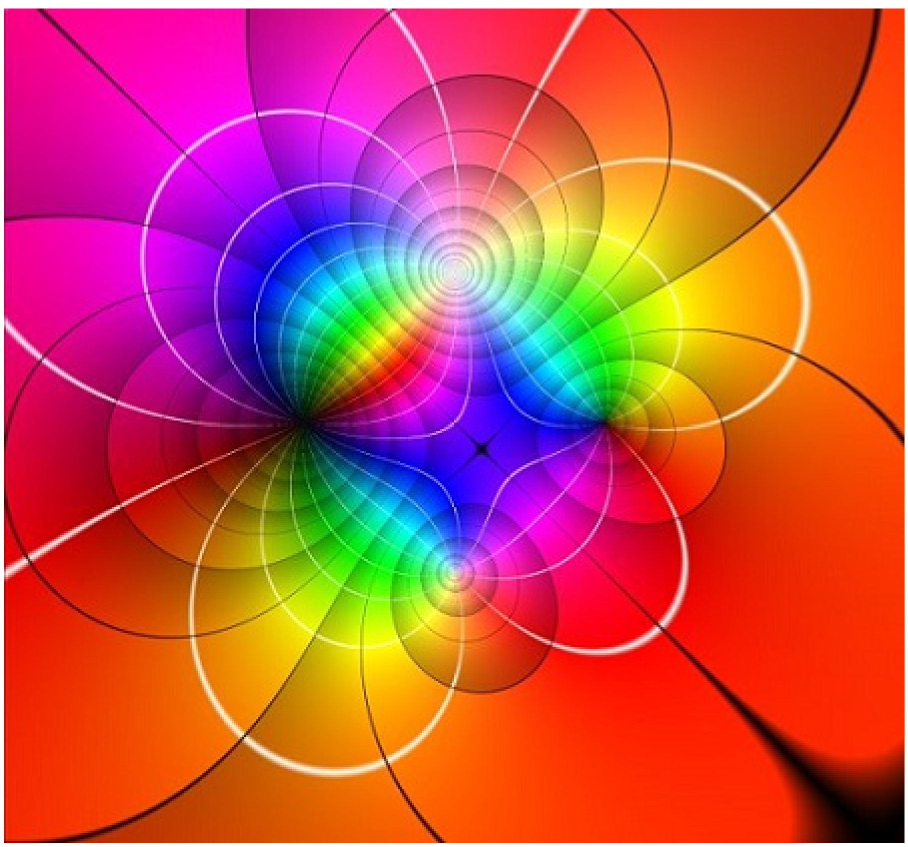


Figure 6: Domain coloring of the Gamma function

If you do a google search of domain colorings, you'll find all sorts of great pictures and software to make these things for you. There are more technical details which one can make note of to make domain colors even cooler looking. Enough talk. Time for cool pictures:



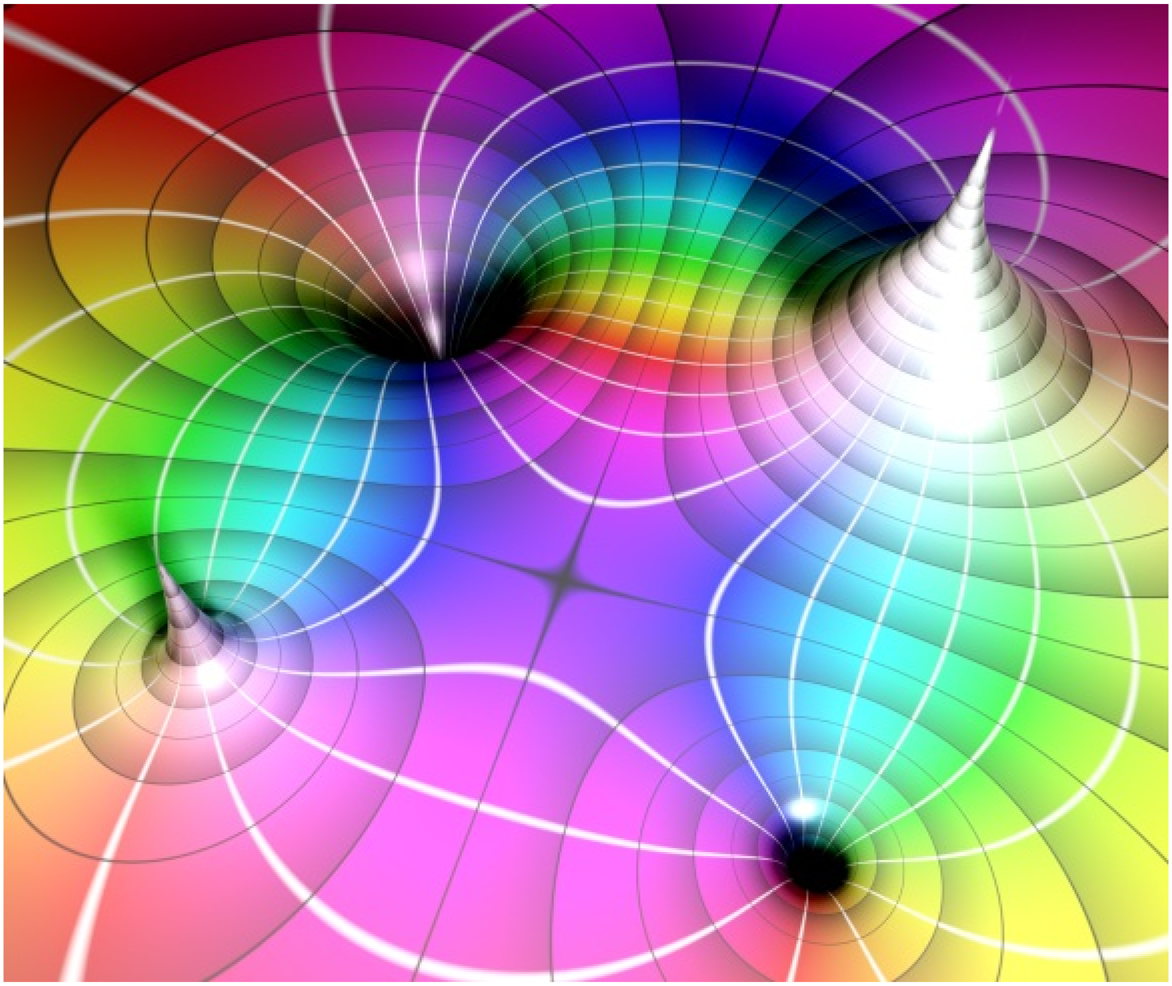


Figure 7: An augmentation called a lifted domain coloring

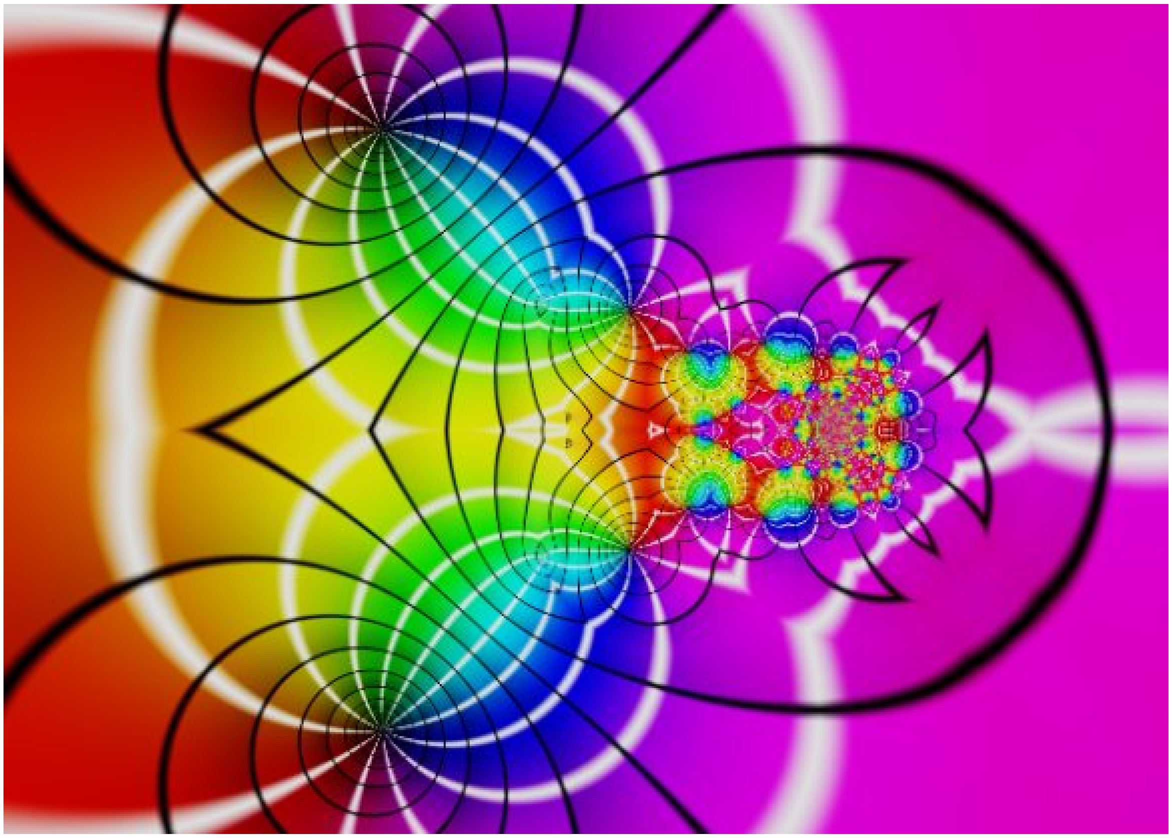


Figure 8: Another by the same guy. His blog name is Syntopia.

$$f(x) = \frac{1}{x} \tag{2}$$