Morishima's Disproportionality Crises and the Contradictions Between Use-Value and Exchange-Value: An Exegesis on Marx's Reproduction Schema

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Abstract

The notion of a disproportionality crisis as a conclusion to Marx's reproduction schema from Capital volume II is nothing new. However, those presentations put forward are for the most part based on Marx's own work in volume II, which is largely agreed upon to be incomplete and deeply flawed in it's execution even on Marx's own terms. In his 1973 work Marx's economics, Morishima proposes a complextion of Marx's reproduction schema based on a correction to his investment scheme. This paper seeks to provide some additional qualitative exegesis on this model. In evaluating his model, we find that the crisis of disproportionality can be seen as a manifestation of the basic contradiction between use-value and exchange-value that Marx outlines in volume I. We additionally note within the model that these crises are distinct and independent of the so-called crisis of overproduction. From this independence the socially necessary of crises within Marx's theory can be witnessed as a conclusion, rather than an assertion. This and more is done in a way which should be more mathematically approachable to readers than Morishima's own presentation, additionally providing an interactive applet with which readers can explore the model visually on their own. Finally, we consider the transformation problem, and takes a cybernetic lens towards a hidden function of money which can be observed within his model: one of obfuscation and information suppression.

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1 Introduction

Of the three volumes of Marx's Capital, it is far from controversial to claim that Volume II is the least read, the least understood, and the least appreciated. Readers often struggle to find the same kinds of insights in volume II that they found in volume I. Most consider the most striking aspect of volume II to be Marx's reproduction schema. However, these are remembered largely due to the method of inquiry serving as a precursor to/partial inspiration for the popular Leontief input-output modelling which came after it, *rather than* due to any particular results which Marx was able to attain from his own inquiry. Marx himself does not seem entirely satisfied with the mathematical results he derived from them, having relatively little to say about his schema in the final pages of the work.

In his book *Marx's Economics*[13], Michio Morishima asserts (far from controversially) that Marx's dissatisfaction with his own lack of results is a consequence of his reinvestment rule. By replacing this rule with something more realistic, and employing more advanced mathematical techniques than Marx would have been capable of in order to derive the dynamics of the resulting system, Morishima arrives at an equilibrium model of capital reproduction and accumulation which, if adequate time is taken to analyze and reflecte on it, add significantly to Marx's argument, not just in volume II but across the entirety of his overall project.

Unfortunately, Morishima's project in *Marx's Economics* is broader than just the reproduction schema, and he is quick to move on to other topics as soon as soon as the difficult math is finished. The result is a presentation which is mathematically dense to point that would be quite difficult for those without significant formal training to grapple with. As a result, it is my opinion that the results found in these chapters have never been given the qualitative reflections that they are truly deserving of. This paper will, through what

I consider a novel approach, attempts to present these results in such a way that *anyone* who has merely a solid understanding of the core Marxist concepts (e.g. surplus value, exploitation, compositions of capital, the industrial reserve army etcetera) as well as a basic high school education of mathematics should be able to understand them without too much trouble. Once this is done, we will, through an interactive online applot available as a companion to this paper, be able to visually explore the resulting model in a way which technology would not have allowed for at the time that Morishima was writing. Within this applet, we will be able to explore not just Marx's reproduction schema but a whole range of concepts significant to his theory, including but not limited to the demand for labor, the transformation problem, and the obfuscating role of money in an equilibrium economy.

With the aid of these tools, we will proceed to reflect on the implications of Morishima's model in a qualitative manner. We will discuss what these results have to say about the fundamental contradictions of capitalism, what insights they have for a Marxist theory of crisis, and how they might have influenced Marx's conclusions both in terms of deriving capitalism's laws of motion and in terms of forming a critique of political economy. To preview, what I will argue is that Morishima's model is really a witness to the fundamental contradiction which Marx began his investigation of capitalism by observing - that between use-value and exchange value - exploding into a theoretical crisis that I call this crisis the crisis of disproportionality. While the idea of a crisis of disproportionality is nothing new to Marxist thinking, dating all the way back to the Russian legal Marxist Tugan-Baranovsky[2], I will argue that the crisis we witness within Morishima's model have a very different character from those traditionally put forward and given the same name in the past. Unlike the disproportionality crises of Tugan-Baranowsky and others, which are primarily derived from the observed infeasability of meeting the balanced growth requirements Marx lays out, and are thus felt through disruptions to the market, the crises of disproportionality which Morishima observes (though does not name as such) occur *independently* of any disruptions to market equilibrium. These crises thus represent a scathing critique of market forces. They show us a clear example of crises arising precisely from market success, as a feature of the market - a breaking point in which the pursuit of exchange-value demonstrably fails to produce desirable outcomes in terms of use-values. I will also argue that, having observed crises which occur both in the presence of market failure and in the presence market success, with or without disruptions to the flow of capital, that this completes the argument which Marx sought to make throughout the volumes of Capital but fell short of doing: crisis is a socially necessary feature of capitalism.

Finally, we will take a cybernetic lens to a particular aspect of the model in order to witness a hidden role of money within the static equilbrium model - that of an information filter, actively hiding any information about the commodities being produced not related to their exchange value. Equilibrium prices under a uniform profit rate thus do not simply allow all capitalists to earn 'their fair share' of profit. They also serve the simultaneous purpose of preventing capitalists from understanding the effects that their market incentives are having on the world of use-values. Marx argued that the transformation of values into prices of production serves to obfuscate and mystify the source of profit away from labor to it's subjects. This observation, which Morishima made mathematically but did not contemplate and which little has been made of since, dovetails nicely with Marx's description of the transformation process. Not only that, it serves as an excellent example of the insights which can be gained once we accept the mistakes in Marx's description of this process, and see the true relationships not as a threat to Marx's legacy, but rather as a phenomenon to be studied scientifically.

To reiterate, all of this will be done without any special mathematical background expected of the reader. There will be no linear algebra present in this paper (aside from a possible footnote or appendix), no mention of spectral theory, eigenvalues or the Perron-Frobenius theorem, and, despite these last points, *no concessions made when it comes to formal rigor*. My goal is to satisfy all readers, regardless of their formal training. These results are important enough that they should be accessible to everyone, and I believe I have accomplished that here.

2 The Dual Character of Marx's Intensions With Capital

Since the framing of this collection of results is as a proposed conclusion to volumne II, we should first try to clearly understand what Marx's intensions actually were in writing it. Only once we understand this clearly can we hope to evaluate whether this conclusion has any validity. Like every other aspect of Marx's thinking, I believe that Marx's motivations here have a dual character.

2.1 Capital as a Scientific Work: Laws of Motion and Social Necessity

The field now known as classical political economy came into being with the goal of analyzing the perceived emergence of what appeared to those theorists as economic 'laws', instantiated through our social actions but nonetheless objective in character.

These new and mysterious 'socio-natural' laws asserted themselves not through enforcement by a state entity, but, rather, like the laws of physics, through their outcomes and effects on people's day-to-day lives. The primary witnesses to these laws were, of course, prices, which while fluctuating according to supply and demand, appeared to be nonetheless orbitting around so-called 'natural prices', which seemed plain to all observers at the time to be reflections of the difficulty of production measured in total resource requirements. Under such a system, in which higher priced goods were shunned by consumers, what Marx dubbed the coercive laws of competition forced capitalists to adopt the most efficient methods of production at the cutting edge of scientific innovation, leading to a uniformizing of the production process in addition to it's increasing level of socialization, in which these observations on the nature of prices could be more easily made.

Marx, in agreement with the classical political economists, believed that capitalism is a mode of production governed not only by state laws but by these socio-natural 'laws of motion'. Free will exists, but is constrained by the reality of these objective economic laws, which, like laws imposed by a state, require an enormous force of will to overcome. Likewise, while the mode of production to Marx is to an extent socially determined, the form it takes is nonetheless severely constrained by the current state of technology and socially accepted methods of production. No phenomena within capitalism embodies this tension better than prices, which on the one hand were believed to be reflections of the socially necessary resource requirements given standard techniques of production, yet on the other hand were seen to clearly vary as a result of the struggle between workers and capitalists over real wage, which is independent of those techniques. We will return to see this tension explode into theoretical difficulty and controversy when we look through the lens of our model at the transformation problem.

Marx's primary goal in the volumes of Capital is to discover and document these so-called 'laws of motion'. In the preface to the 1867 German edition of Capital Volume I, Marx says: "...it is the ultimate aim of this work, to lay bare the economic law of motion of modern society".([7])

We will not dwell here on the details of Marx's dialectical method here. What is important is merely to establish that Marx does *not* merely see his work as that of critique, despite what the subtitle says and what many modern Marxologists tend to place emphasis on. Marx sees himself also as a scientist, attempting to derive the laws of motion of any capitalist system, investigate the consequences of these laws, and test those laws against the real course of history.

Lastly, it is not just laws of motion which Marx intends to discover. Repeatedly Marx shows interest in the notion of *social necessity.* By this, Marx is referring to consequences of the laws of motion - effectively theorems - general facts which *must always* be true of *any* capitalist system, regardless of the historical particularities. Marx's goal is to demonstrate that many of the dynamics of any capitalist system can be seen as witnesses to general laws which follow from first principles of a capitalist system; the basic requirements of what is necessary for such a system of production to exist and reproduce itself. This reproduction involves not just the reproduction of the materials but also the class relations themselves. Before looking more closely at this, however, we turn to Marx's second major intension.

2.2 Capital as a Critique of Political economy

While Marx sees his primary goal as deriving laws of motion, he is building his observations on top of the work of the early political economists of his time period. Marx believes that the scope of observations made by these political economists is quite limited, and has many criticisms and corrections to make of these thinkers as he investigates the capitalist mode of production. This critique has two core dimensions.

First, Marx believes that the classical political economists have not done a sufficient job in understanding the nature of the natural laws they are observing. In particular, *how* these come into being, and furthermore what the ideological effects of these laws are on the subjects of the system. To put it in a more modern sense, what Marx sees as missing from these accounts is a sufficient consideration of *feedback* and *feedforward*. Marx understands social systems as feeding back on the individual, effecting them on a deep level that alters their perception of themselves. How does something as anthropologically bizarre as the value-form come into being, in which any object is equatable in an objective manner to some number of any other object, and how does this effect the ideologies of the participants in such a society? How does it come to be that one group of people possesses means of production while the other is left with no way to survive other than selling their labor power? What brings the capitalist to the market in order to purchase this labor power, and how does the class antagonism between these groups effect their political beliefs?

Second, while it would likely be unfair to call all of the political economists utopian in their thinking with respect to what capitalism's natural development would lead to, it is fair to say that they were mostly in agreement on one front: capitalist society, and in particular market mechanisms, would serve to coordinate labor and distribute resources in a way which, while not serving to perfectly meet the needs of it's participants, would nonetheless distribute resources *optimally*, so as to meet the shifting wants, needs and desires of the population to the maximum extent possible given the current social productive capacity and scarce resource constraints. In short, they saw capitalism as 'the best of all possible worlds'.

Marx intends to contest this idea. To quote David Harvey: "*Marx is engaged in a critique of classical liberal political economy. He therefore finds it necessary to accept the theses of liberalism (and, by extension to our own times, neoliberalism) in order to show that the classical political economists were profoundly wrong even in their own terms. So rather than saying that perfectly functioning markets and the hidden hand can never be constructed and that the marketplace is always distorted by political power, he accepts the liberal utopian vision of perfect markets and the hidden hand in order to show that these would not produce a result beneficial to all, but would instead make the capitalist class incredibly wealthy while relatively impoverishing the workers and everyone else."*([4] p.52)

One of the great tensions felt throughout the volumes of Capital is between Marx's acceptance of many of the premises of the political economists for the purposes critique, and his intention of identifying actual economic laws that he himself believes in. This tension has been the source of a great deal of controversy over the years in the interpretation of Capital. To again quote David Harvey on the matter: *"We have to be careful to distinguish between when Marx is talking about and critiquing the liberal utopian vision in its perfected state, and when he is attempting to dissect actually existing capitalism with all of its market imperfections, power imbalances and institutional flaws. As we will see, these two missions sometimes confound each other. Some of the muddles of interpretation come from this confounding."*([4] p.53).

My personal opinion here is that this tension is not as pronounced as it has often been made out to be. Marx's critiques, in my opinion, are rarely the premises themselves, and I do not believe that Marx at any point accepts a premises that he does not believe to be valid on *some* level. For example, I've seen it claimed by extremists of the critique viewpoint that Marx does not even actually have a labor theory of value - that he merely is accepting this premise on the origin of prices from the political economists themselves. This is absurd. While it is indeed the case that Marx's labor theory of value is partially inherited by the political economists, he is adopting it because he agrees that they got this aspect of their analysis right. The critique comes *after this*, either in the details of how the theory is implemented, or in the the flawed foundations of their premises, or from flawed in the conclusions drawn from the premises, etcetera.

2.3 The Premise of an Equalized Profit Rate

The premise Marx accepts which is most relevant for us in this paper, and by far the most controversial, is that of perpetual supply-demand equilibrium, or more particular a hidden requirement for that equilibrium: an equalized rate of profit across all industries. Without an equilibrium rate of profit, perpetual supply-demand equilibrium is impossible, as even if we started our system in such a state, capitalists would immediately begin shifting production towards the more profitable sectors, disrupting said equilibrium.

It does not take much effort to understand how impossible an equilibrium system of this sort actually is. First are the logical issues. If supply and demand are in perpetual equilibrium, then price signals provide no information to the capitalists. This is a problem because changes in price are supposed to be what inform the capitalist on what to produce and how to adjust their production in order to meet demand and maintain said equilibrium. Supply-demand equilibrium is paradoxical in that once we assume it, we immediately lose the mechanism by which it was supposed to be maintained. Then there are the empirical and theoretical issues. While there are at this point theoretical models which demonstrate a tendency for profit rates to converge[18], as well as modest empirical evidence that the tendency is at least felt[5], none would argue that such a process of profit rate convergence has or would ever complete in a real capitalist economy. The assumption of a convergence towards uniform profit rates would be even less plausible from the perspective of the political economists of Marx's time, who invented the notion, but had no theoretical or scientific evidence to motivate the belief. Before we ask why Marx himself accepted this premise then, we need to think then about *why* the classical political economists believed in such a notion.

As we already noted, these 'utopian' political economists strongly believed that market forces would effectively force capitalists, through competition, to adjust production in such a way that the use-values produced in capitalism would always roughly conform to the actual wants, needs and desires of the consumer. It is important to recognize that this belief in market forces is precisely what resolves the contradiction between use-value and exchange-value which Marx observes at the very beginning of Capital volume I.

In capitalism, capitalists are motivated to produce by the incentive of acquiring exchange-value, or money. However, the mandate of a mode of production is to create *use-values* - useful things which society needs in order to survive and thrive. The contradiction is simply the observation that the ones deciding on what to produce have no direct incentive to address the use-value needs of society. To the extent that capitalists produces satisfactory outcomes in terms of use-values, this is only ever *effective*, or *accidental*, and never directly their intent. According to the classical political economists, as well as modern day proponents of capitalism today, these market forces are precisely what force the pursuit of exchange-value to also produce desirable use-value outcomes. The classical political economists were smart people, however. They understood that *non*-uniform profit rates severely undermined their narrative, as their presence represented a factor which would steer production *independently* of people's wants, needs and desires. My argument is that the assumption of a convergence towards equilibrium profit rates was one of motivated reasoning, in order to complete their story. As long as profit rates equalized, they believed, their story was true, and the correlation between the pursuit of exchange-value and desirable use-value outcomes would be cemented.

This leads us to Marx's adoption of the premise. Firstly, as already stated, I am of the belief that Marx would not have accepted this if he did not believe there was something 'real' about the scenario. While profit rates can never be expected to fully equalize, an equalized rate of profit could (and, at least theoretically, does) still exist, and to the extent that it does, it still acts as an attractor for the dynamics of a real capitalist system, perpetually exerting a *real* force which pulls the system towards this theoretical scenario[20]. Furthermore, while of course it is impossible to maintain supply demand equilibrium perfectly, it is very reasonable to consider a situation in which capitalists respond very quickly to price signals and shift their production accordingly, thus hugging close to whatever dynamics are observed in the simpler case of perpetual smooth equilibrium. Combined, this 'steady state' capitalist system, even if only existing in a purely theoretical sense, exerts real consequences on the actually existing capitalist system. However, more than any of these 'physical' justifications for accepting the premise, I think that Marx above all accepts this premise because he believes that he can contest the premise that such an economy produces the 'utopian' outcomes that the political economists assumed it would.

To summarize and conclude this discussion of why Marx adopted the premise of an equilibrium rate of profit, and why even today such a premise is still worth considering, we should note that the political economists were correct in their motivated reasoning: if there is any scenario in which market incentives alleviate the contradictions implicit to capitalist production between use-value and exchange-value and generate prosperity, then we should be able to observe this prosperity within a theoretical model which assumes an equalized uniform rate of profit. *Contrapositively, if we cannot, then we must conclude that the story told to us by the classical political economists was wrong.* Marx was not able to arrive at such a bombshell result, but as we shall see, he was very close - close enough that all we need to do in order to arrive at it ourselves is finished what he started.

3 Social Reproduction

As we noted in section 2.1, Marx is interested in obtaining general 'theorems' (which he calls 'social necessities') of the capitalist system - laws of it's dynamics which are true regardless of any historical particularities. Marx's primary method of obtaining such results is from investigating the question of social reproduction.

Here, Marx is interested in analyzing how capitalism reproduces the conditions of it's own existence. His hope, and indeed his great insight, is that by doing so - by carefully determining what is socially and logically necessary for this reproduction to occur - he will be able to identify contradictions within this process, and through analysis of those contradictions determine the generalities of how any such system develops over time.

Both volumes I and II culminate in a discussion of different aspects of this question of reproduction. Volume I, focused as it is on production, reaches it's climax through a direct inspection of the reproduction of the class relations themselves. It is here where Marx obtains his laws of the reserve army of labor, and his brilliant general law of capital accumulation. Marx caps off volume II with a return to the question of social reproduction, but this time rather than focusing on the reproduction of the social or class relations between peoples, Marx is interested in the reproduction of the *material relations between capitals*. Marx is interested is how the vast interdependent network of capital circulations, each aiding and reinforcing each other, manages to reproduce itself. To capital, labor is just another resource, no different than the means of production or any other input necessary for its renewed circulation. The concern here therefore is merely the flow of goods and their availability in the market.

Toward this end, Marx makes a collection of very clear and explicit assumptions in both inquiries in order to facilitate his focus on one of these aspects of reproduction at a time. In effect, when he is focused in volume I on reproduction of the social relations between people, Marx's assumption amount to assuming no problems in the reproduction of the material relations between things. More specifically, he assumes two things:

(1) That all means of production which the capitalist might desire always appear in the market, available for purchase at their natural prices, which Marx assumes for simplicity is equal to their labor values.

(2) That all goods produces by the capitalist will find a buyer in that market at the end of the productive cycle.

In these two assumptions we can see the *entirety* of Marx's concern in volume II. Marx makes them in order to focus his attention on the reproduction of the capital-labor relation and idealized dynamics of the reserve army or labor.

In Volume II, Marx fully inverts this lens. In particular he assumes there exists an inexhaustable pool of available labor available for purchase at the natural value of labor power, so that he can focus on the dynamics of how capitalism manages the circulation of materials in an interconnected system of productive capitals. In other words, how, if at all, does capitalism go about reproducing or approximating those conditions (1) and (2)? What is necessary for such reproduction, and what contradictions can be identified within those requirements? We now turn to briefly describing how Marx goes about this.

4 Overview of Marx's Reproduction Schema

4.1 Capitalism as a Circulatory System

A single capital social relation is self reproducing, but this reproduction is contingent on the availability of necessary inputs: means of production and labor power. The means of production are primarily commodities which produced as the residual outputs of other capitals. In order to analyze these many flows and make them manageable, Marx breaks his capitalist economy into two broad 'departments'. Department I is the collection of all industries producing means of production (capital goods), while department II is the collection of all industries producing consumable products (wage goods). This choice is not arbitrary. Rather, it is perfectly consistent with his breakdown of value itself into value in means of production (C) and value in living labor ($L = V + S$). Implicitly, these two departments partition the set of commodities produced by a capitalist system into two distinct use-value categories of goods.

Figure 1 is a high level graphical depiction of how Marx conceived of the capitalism as an overall circulatory system. This conception is inspired by the french Physiocrats, in particular François Quesnay, a physician who based his understanding of economics on his understanding of blood flow in the human body. To Quesnay, agricultural products took the place of blood, as the material which provides life by way of it's continual flow. Marx adopted Quesnay's approach but with with the crucial correction of replacing the

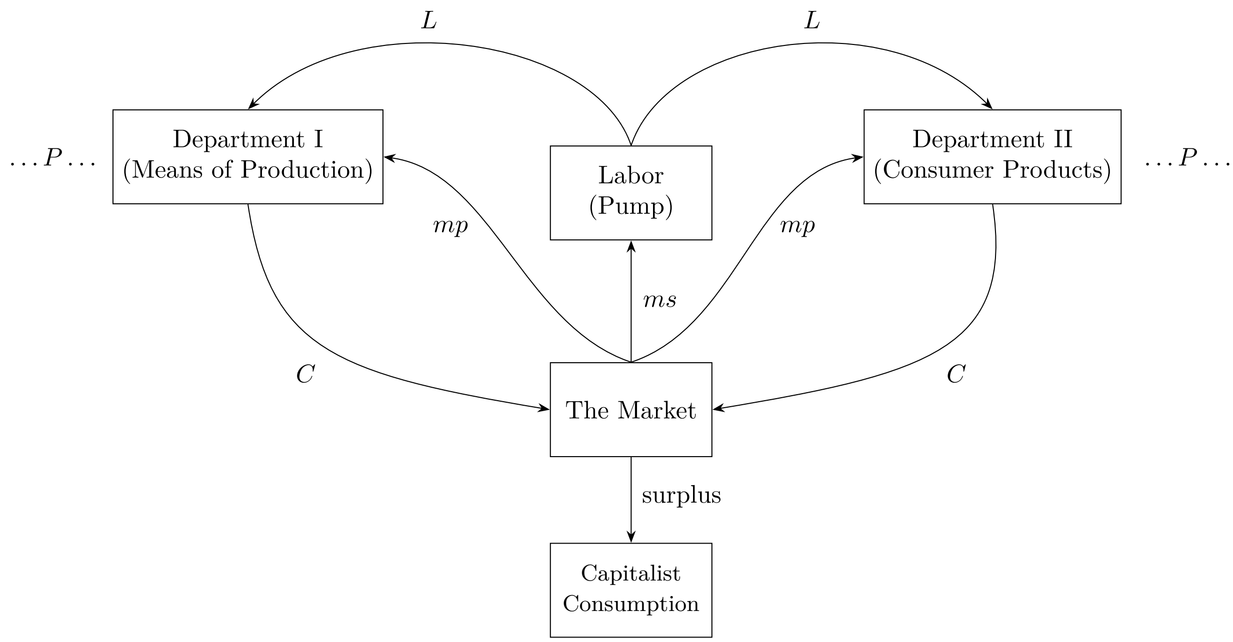

Figure 1: Marx's conception of the capitalist economy as inspired by the French physiocrats.

flow of agricultural product with the flow of labor.[1] We see in the figure a labor 'organ' acting as the heart, 'pumping' labor power into the two departments, where production occurs. Flowing outward from these departments are finished commodities which are sent to market and redistributed to the departments. The market therefore acts as a central routing hub of sorts: all goods first flow into the market, from which they can then be distributed - means of production to the departments, and consumer goods to the workers and capitalists. This diagram chooses to depict specifically the 'real' flow of use-values through the system. It should be noted that facilitating each these flows of commodities are flows of money, in the opposite direction. These are omitted only for simplicity.

From this, we go one step further, and break the circulatory network down into three distinct and independent classes of circulations, depicted in in figure 2. I've chosen to label these three macro-circulations which I've labelled α, β, and γ. (The reader is encouraged to trace these circuits in the original diagram. They will find in doing so that all lines are accounted for except for the terminal arrow to the capitalist households from the market, which I added only for completeness.) Circulations α and β represent the two *intra*departmental circulations. α is the circulation of means of production which stay within the department itself - e.g. means of production to be used in the production of more means of production. β is the circulation of consumer goods which are consumed by workers employed in the consumer goods industries. These goods are intradepartmental insofar as they are produced by workers and then consumed by those same workers.

The remainder of the capital goods produced by department I which are not part of circulation α must go to department II, represented by the top arrow of γ. Likewise, the remainder of consumer goods produced by department II must be used to feed workers in department I, and this is represented by the bottom arrow of γ. Hence, γ represents the *inter*departmental circulation of goods within the economy. Note that this is only circulation if we allow for the transformation of goods into money. The two flows of commodities are different, and are only equal in terms of their values.

This breakdown of capitalism as a system of three broad circulations of goods structures Marx's analysis. Marx proceeds to inspect each circulation thoroughly in the context of both simple and expanded reproduction. It is specifically in his inspection of the interdepartmental circulation γ that his analysis stumbles, and so we will begin with brief summaries of his observations on the first two intradepartmental circulations,

[1] Quesnay himself broke society down into the 'proprietar' class consisting of landowners, the 'productive' class consisting of agricultural workers, and the 'sterile' class consisting of artisans and merchants. This is understandable given his experience living in France, which was overwhelmingly still an agricultural economy at the time.

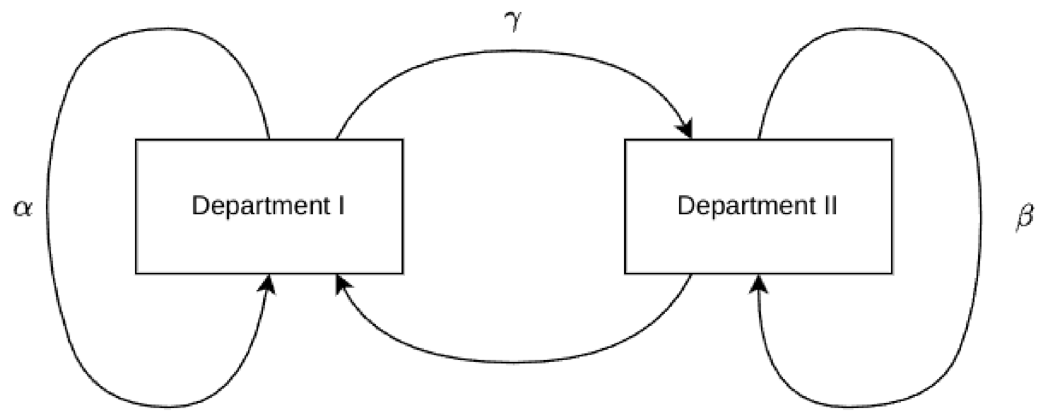

Figure 2: The three major circulations of interest. See if you can identify the circuits that each of these stands in for in the previous figure.

before proceeding to the main attraction.

Before we get started, at this point we must address a crucial assumption which Marx makes in his analysis here: that our capitalist society maintains continual supply-demand equilibrium at all times. Marx is doing this not necessarily because he believes such a thing is possible, but because he is trying to critique the political economists of his time, who themselves analyzed capitalism under ideal conditions in order to diagnose its viability as a system capable of producing stable prosperity and growth for all people. Today, equilibrium models such as this are seen as much more questionable, and this is for good reason. But those objections do not need to concern us (or Marx) here, at least not yet. Whether such equilibrium economies can possibly exist in real life or not (they cannot), the important thing is to note that this *should* be precisely the situation in which markets and capital can perform at their best. If the outcomes observed in such a 'steady state' equilibrium system turn out to be undesirable, then regardless of whether such a system is possible or not, this would be very bad news for anyone who would advocate for free markets, and for actual existing economies which in any way approximate perfect market dynamics.

The assumption of perfect supply-demand equilibrium here is important to Marx's analysis because it determines the fundamental question he intends to ask as he looks at these three circulations: what is socially necessary for such a system to continue?

4.1.1 Circulation α: Problems of the Supply Chain

Looking at α, Marx notes that this circulation involves *all* possible supply chains, since by definition end-products (produced by department II) are never going to be inputs to the production of other commodities. Marx notes that this circulation involves a disproportionate amount of fixed capital, which is characterized by long and varied turnover rates. Fixed capital is also expensive, in the sense that it tends to require a large sum of capital *advanced*. Thus Marx observes that capitalists in department I will be forced to hoard a portion of their capital in order to be ready to afford to replace it when that time comes. Hoarding is antithetical to capital, as it represents a blockage in the overall flow of social activity. Thus Marx arrives at the basic contradiction implicit to circulation α: On the one hand, capitalists must repeatedly pause their individual flows of capital by hoarding money to pay for replacement of their fixed capital. On the other hand, the fixed capital being hoarded for are essential links in the supply chains of other capitalists. This creates the danger that capitalists might find themselves held up waiting for capital inflows being which are stuck being hoarded by other capitalists. It is thus here in circulation α that the contradictions of fixed and circulating capital naturally rear their head, revealing a latent potential for supply chain disruption. Like most contradictions Marx identifies throughout the volumes of Capital, there is no guarantee that crises of the supply chain will result from this observation. Rather, Marx is observing a tension implicit to how capitalism must operate. These tensions are useful to observe in that they allow us to understand causally many otherwise disparate developments in the history of capitalism. Here, for example, we can understand mass adoption of just-in-time production techniques.

4.1.2 Circulation β: Problems of Overproduction/Underconsumption

Turning to circulation β, Marx observes observes that unlike circulation α, the health of circulation β depends on continual rational consumption on the part of the working class. Whatever the basket of goods composing the means of subsistence happens to be, the ability for these capitals to renew themselves is contingent on workers purchasing and consuming goods in the way that capitalists want or expect them to. Additionally, just as circulation α had a hoarding problem on the part of the capitalists, circulation β has a hoarding problem on the part of the workers, since the decision to save money is the decision to not spend it and renew the circulation of capital. Capitalists are therefore going to have a tendency to hate seeing their workers accumulate savings. They are also going to have a tendency to obsess over their worker's spending habits. Perhaps most interestingly, they are going to want to see workers spending their wages as quickly as possible, in order to minimize turnover times. Thus, where contradictions between fixed and circulating capital found their representation in circulation α, it is β where we can observe all manner of contradictions of worker consumption.

4.2 Circulation γ: The True Object of Interest

This third circulation - the interdepartmental circulation - is Marx's primary concern among the three. It is clearly the most delicate of the three, and it is equally clear that Marx believed he could find something of significance here by singling it out. Let's look at how he went about this.

4.2.1 Simple Reproduction

Again, Marx is primarily concerned with asking the question: what is socially necessary in order to maintain continual supply-demand equilibrium between the two departments? It is in circulation γ where the requirements become most delicate, and most interesting. To start, Marx seeks to determine the conditions for maintaining equilibrium between the two departments. Let Y_1 and Y_2 denote the total value output from departments I and II respectively. As we did with society as a whole, we can divide these value outputs into C, V and S components.

$$Y_1 = C_1 + V_1 + S_1 \tag{1}$$

$$Y_2 = C_2 + V_2 + S_2 \tag{2}$$

To clarify, what we have in Y_1 is the sum total value of all commodities produced by department I at the end of a production period. Consider figure 2. Each of the arrows of this diagram represents simultaneously a flow of real goods, but also a flow of value. We are assuming simple reproduction, meaning that no value is accumulating. In other words, the total value circulating within the arrows of figure 2 must be equal to $Y_1 + Y_2$. Our task is to see where each of the six components belongs.

Let's begin with C_1. This is therefore the total value of all means of production - all dead labor - used by department I. If department I used this total value in producing Y_1, then in the context of simple reproduction this is exactly the amount that they will need to use in the next production period. In other words, the value circulating through circulation α is exactly C_1. Since there is only one more outgoing arrow from department I, this means that the amount of value moving from department I to department II is precisely $V_1 + S_1$.

Turning next to the output from department II. In particular, let's look at V_2 and S_2. V_2 is an output, but it is also the total wages paid to workers in department II. Again constrained as we are by the assumption of simply reproduction, it follows that workers must spend exactly this amount in purchasing consumer goods. Additionally, simple reproduction requires that capitalists not be reinvesting, and thus they must be consuming all of their surplus. Meanwhile, S_2 is the surplus value produced by the capitalist in department II. It follows then that they must be consuming exactly this amount of their own departmental output. Since this accounts for all consumption of workers and capitalists involved in department II, we have our answer for the total value of circulation β: $V_2 + S_2$.

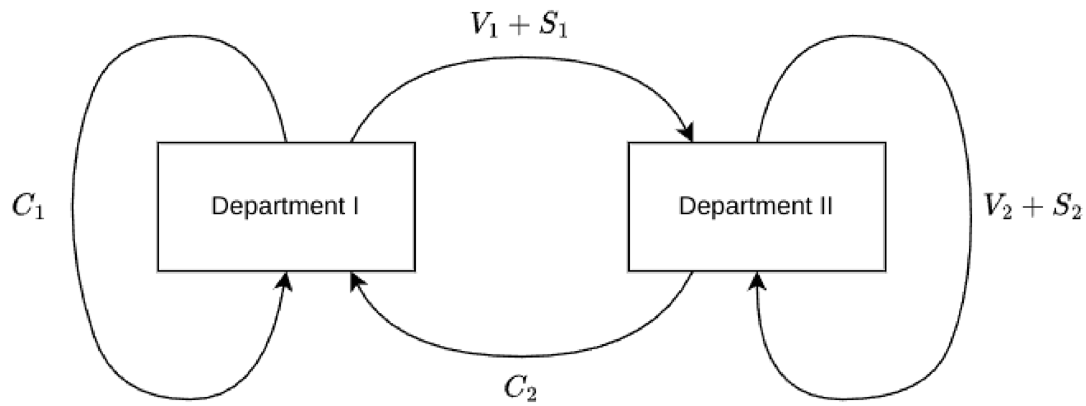

Figure 3: The three major circulations of interest

This leaves the remaining C_2, which only has one place it can go, just as we say for $V_1 + S_1$. We therefore have that the value flowing from department II to department I is C_2. Figure 3 displays our system with these amounts labelled.

By making these observations Marx obtains his equilibrium condition for capitalism under the special case of simple reproduction. To put it bluntly, the two 'rest-of-its' which we found have to equal one another. Expressed mathematically, we have the condition:

$$V_1 + S_1 = C_2 \tag{3}$$

What Marx does next is inspired by Quesnay's *Tableau économique*. With that as a basis, he attempts to work out a concrete numerical example of flows between the two departments which facilitate an expanding economy. The numbers and calculations in the case of simple reproduction are not important to me, and there is nothing wrong with them. The exact numbers and calculations used in the case of expanding reproduction are where the problems lie, but even these are not as important to me as his conceptual set up. The numbers I mention in this summary will only be for the sake of pointing out issues which Marx may or may not have been aware of, and which our own model will need to account for. Later, towards the end of this paper when we are in a better position to understand it, we will look at his example more closely.

4.2.2 Marx's Approach: Expanded Reproduction

Equipped with this equilibrium condition, Marx moves on to the case of expanded reproduction - i.e. the case in which capitalists are reinvesting their surplus and expanding the scale of their production. There are multiple ways in which an economy can go about growing. Capitalists have a choice not only of how much surplus to choose to reinvest, but also the choice of *where* to invest it. A growing economy could favor department I over department II, for example. Marx makes his intentions clear on how he wants his capitalist economy to grow up front: *Now let us analyse scheme a) more closely. Let us suppose that both I and II accumulate one half of their surplus-value, that is to say, convert it into an element of additional capital, instead of spending it as revenue.* [9](p. 582)

This quote expresses Marx's clear initial intention for what he wanted his model to do. He wanted to have capitalists from both departments investing a fixed proportion of their surplus value; in his specific example, half. This is reasonable. However, *it is not at all what he proceeds to do.* In actuality, Marx's proceeding numerical work has it being the case that at the end of each production period, capitalists from department I reinvest half of their surplus value, and *then, only after this, capitalists from department II would reinvest exactly the amount necessary to allow for this* to be possible.

This is quite a strange choice, for a variety of reasons, not least of which being the fact that it flies in the face of what he said he was going to do just pages earlier. It also makes no sense as a realistic model of how the capitalist system works or how capitalists actually behave. Capitalists in department II are not simply going to wait for capitalists in department I to make their investments first before reacting. (This makes even less sense when we realize that many of these capitalists likely have investments in both departments, and are thus bifurcating their own investment behaviors based on where their profits came from.)

Much has already been said on the matter of this strange choice of this reinvestment scheme. I don't think it really needs to be explained for readers to understand the unsatisfactory nature of having capitalists in department II being effectively slaves to the capitalists in department I. To quote Morishima on the matter: *The un-natural adjustment of the rate of accumulation by capitalists of department II to the exogeneously determined rate of accumulation in department I was invented by Marx merely as a deus ex machina. However we should not be too surprised that Marx performed so poorly in this case. Even Walras could not properly solve the simultaneous differential (or difference) equations describing the process of tatonnement. Remembering that Marx had begun his academic career as a philosophical student and learned mathematical economics by himself, we should be greatly impressed by his model, which may be taken as the prototype of the present day Leontief-von Neumann models. In mathematical economics, like other sciences, the most important thing is to pose fundamental problems.* ([13] p. 125)

However, we must again note that Marx *did* initially pose the problem correctly. It is unclear to me from my own reading whether or not Marx was consciously aware that the numerical example he had worked out wasn't the reinvestment scheme he originally proposed. Our completion of Marx's work will be to work out his original idea: to creating a growth model in which capitalists from both departments are reinvesting half of their surplus value. We haven't gone through the details of Marx's calculations yet - we leave this to later in the paper. I believe at that point it will be easier to evaluate.

5 A Gentle Derivation of Morishima's Solution

I hope I have convinced the reader what 'fixing' Marx's reproduction schema means here. Our task is to find equilibrium growth paths for a capitalist system in which capitalists from both departments are investing a fixed proportion of their surplus value back into production each period. Additionally, we must maintain supply-demand equilibrium at all times. Such conditions are what the liberal political economists of Marx's time claimed would be prosperous to all. After deriving these paths, we will try see if there is anything wrong with that outlook. (It turns out that this part will not be difficult.)

My presentation here is meant to be understandable by anyone regardless of their mathematical background. For those more mathematically inclined, finer details and the process of actually solving the equations we set up here can be found in appendix ???. As for analyzing the results, I have included with this paper an 'app' which can be used to visualize the solutions to these equations, and I strongly believe that what the reader sees in that visualization will be enough to show them clearly how thoroughly it shatters the arguments of the political economists of Marx's time, as well as what it says about free market capitalism in general. If the reader gets lost in this derivation, they are highly encouraged to return and reread it after reading our discussion of the solutions.

5.1 Key Variables and Assumptions

First, let us state the assumptions of our model capitalist society:

1. The technological state of society is fixed and unchanging. As we will see, this does *not* necessarily mean that the overall composition of capital will remain the same from one production period to another. Rather, it means that the individual compositions of capital of each department are fixed and unchanging. If a capitalist from the steelmaking industry generally spends 80% of their capital on means of production and 20% on labor power, then the composition of capital for steelmaking is $\frac{80}{20} = 4$, and *this* is what we are saying will not change.

2. The rate of exploitation is a global social constant. Two capitalists who purchase the same amount of labor power will always receive the same amount of surplus value for their purchase, and this amount will never be allowed to change.

3. The money rate of profit experienced by capitalists is equalized across all industries. Commodities always sell at their fair prices of production.[2]

4. Since we are concerned with the reproduction of capitalism's circulatory system, we will assume, just as Marx does, that all is well in terms of the reproduction of capitalism's labor relations. By this we mean that the industrial reserve army will always be sufficiently large that capitalists have no problem in finding workers to hire when they want to scale up their production. We further assume that the reserve army is large enough that no amount of hiring or firing within our society will ever be on a scale large enough to significantly affect the pool, thus stabilizing wages.

5. No value produced is ever hoarded by either class from one period to the next. All value produced during a production period is either consumed or reinvested before the start of the next period.

In addition to these assumptions we also have requirements. In particular, we require that at the start of a given production period, all capitalists will choose to reinvest the same fixed proportion of their surplus value, and that supply-demand equilibrium is maintained at all times. This statement is precisely what we must convert into mathematics.[3]

Towards this end, let us begin to define our variables. We'll begin with the accounting identities we wrote down when discussing Marx's simple reproduction schema: let y_1 and y_2 represent the total value output from the two departments (that is to say, the total combined value of all commodities produced by each department during the previous production period). In the basic Marxist accounting scheme, we can break both quantities down into constant, variable, and surplus components:

$$y_1 = C_1 + V_1 + S_1 \tag{4}$$

$$y_2 = C_2 + V_2 + S_2 \tag{5}$$

In the context of simple reproduction, the numbers y_1 and y_2 remained the same each period. All value produced during a production period was consumed before the next; none is hoarded or invested. Now however, these numbers are expected to change from period to period, and so we see them as functions of time. Let $y_1(t)$ and $y_2(t)$ denote the total value output by departments I and II respectively at the beginning of production periot t (or, equivalently, at the end of period $t-1$). $y_1(0) = y_{1i}$ and $y_2(0) = y_{2i}$ will denote the initial conditions for our system.

In the new context of extended reproduction, equations 4 and 5 are not going to be sufficient in terms of breaking things down into C, V and S components. The reason for this is that if our y values are changing, then the associated C, V and S values will be changing as well. However, there is still something constant about the proportions in which y_1 and y_2 break down into these components which we don't want to lose track of. Isolating these invariants will make our lives much easier.

To keep things as down to earth as possible, let's work our way up from first principles to obtain these constants. Consider a particular commodity, say some model of car. There is a specific bundle of materials which is needed to produce the car, and that this bundle is the same for all manufacturers of that car. That bundle has a fixed value, say of 300 hours (the unit of time we use to measure value is unimportant - we will go with hours for convenience). As we mentioned, this value can be broken down into it's living and dead labor components. Suppose that 200 of these 300 hours constitutes the value of the means of production consumed in making the car (i.e. the dead labor). The remaining 100 is the time used to assemble the car from those materials. This $2:1$ proportion applies to the car industry generally. Given a single *hour of output value* from the car industry, we will always have that $66\% = \frac{2}{3}$ of that hour is the value of materials used, and $33\% = \frac{1}{3}$ is the value of the living labor added on. The shift here is in our thinking about

[2] We will not need to concern ourselves with what the equilibrium profit rate experienced by capitalists actually ends up being in the derivation of our model. However, our model's parameters will end up fully determining what the equilibrium profit rate has to be. Thus the transformation problem will eventually rear it's head within our model, but we do not need to concern ourselves with it right now. We will look at this more closely in section 7.

[3] It should be noted that there is a discrepancy here, in that we are having capitalists make their reinvestment calculations in terms of their surplus value, even though the transformation of value into price has obfuscated and hidden those numbers from them. We will demonstrate in section 7 that having the capitalists reinvesting their profits instead of their surpluses changes nothing about our observations.

output from the car industry. Instead of thinking about cars as the unit of output, we are thinking about an abstract hour of output value, and not concerning ourselves with whether or not this corresponds with a single commodity. This is useful, as it homogenizes the outputs of disparate industries and allows us to more easily discuss them together. Define $c = \frac{2}{3}$ and $l = \frac{1}{3}$, and note that these numbers characterize the technological state of our car industry. Every industry has their own little c and little l, and all of these c's and l's obey the identity that

$$1 = c + l \tag{6}$$

Next, define c_1 to be the *average* of all c's belonging to industries in department I, and c_2 to be the same for department II. Likewise, define l_1 to be the average of all l's belonging to industries in department I, and l_2 to be the same for department II. It can easily be shown that $c_1 + l_1 = 1$ and $c_2 + l_2 = 1$[4] The above steps might have seemed small and tedious, but I believe this step is worth doing carefully because it marks an easily missable point at which we are transitioning from our microeconomic assumptions (that these numbers exist and are constant for each industry) to a macroeconomic model which is approximating our *actual* (but still theoretical) model economy. [5]. I'll leave it to the reader to judge for themselves whether or not this fundamental approximation meaningfully calls any of the conclusions we reach into question.

We can now apply these constants to our output functions. Given $y_1(t)$ hours of output value from department I, we have that $y_1(t) = c_1 y_1(t) + l_1 y_1(t)$, i.e. we can now always compute what portion of a given amount of output from the department is value in the means of production used to produce the output and what portion of it is value in living labor.

Next we bring into the discussion our rate of exploitation. We will denote this number e. Recall that this is to be fixed and constant in our capitalist society. Consider the l_1 hours of labor from department 1. Of this amount, some of it was paid for by a capitalist, and the rest was given over as unpaid labor. The former amount we will denote v_1 and the latter we will denote s_1. We similarly define v_2 and s_2 for department II. Thus we have

$$l_1 = v_1 + s_1 \tag{8}$$

$$l_2 = v_2 + s_2 \tag{9}$$

While v_1 and s_1 may differ from v_2 and s_2, our assumption of a uniform and constant rate of exploitation requires that the *ratios* be equal. That is:

$$\frac{v_1}{s_1} = \frac{v_2}{s_2} = e \tag{10}$$

Finally, we have the departmental rates of profit[6]:

$$\pi_1 = \frac{s_1}{c_1 + v_1} \tag{11}$$

$$\pi_2 = \frac{s_2}{c_2 + v_2} \tag{12}$$

We now have a full system for breaking down arbitrary value outputs from the two departments, using the six technological c, v and s constants. These allow us to define the departmental (technical) compositions of capital k_1 and k_2:

$$k_1 = \frac{c_1}{v_1} \tag{13}$$

[4] Simply note that

$$c_i + l_i = \frac{\sum c}{n} + \frac{\sum l}{n} = \frac{\sum c + l}{n} = \frac{\sum 1}{n} = \frac{n}{n} = 1 \tag{7}$$

for $i = 1, 2$.

[5] For example, it allows us to note that the accuracy of our model depends on how close the compositions of capital for the individual industries of each department are to the average, e.g. that the statistical variance of the compositions composing each industry is small. Having this not be the case does not necessarily invalidate the predictive accuracy of the model, but there would be more to show in order to prove that it does not

[6] Note that these two profit rates will not have any direct bearing on the dynamics of our model, because we are assuming a that a uniform profit rate prevails. Nonetheless, they are still theoretically noteworthy.

$$k_2 = \frac{c_2}{v_2} \tag{14}$$

To summarize, we have the identities:

$$y_1(t) = c_1 y_1(t) + v_1 y_1(t) + s_1 y_1(t) \tag{15}$$

$$y_2(t) = c_2(t) + v_2 y_2(t) + s_2 y_2(t) \tag{16}$$

$$c_1 + v_1 + s_1 = 1 \tag{17}$$

$$c_2 + v_2 + s_2 = 1 \tag{18}$$

5.2 Warmup: Ideal Growth Without Resource Constraints

Suppose for the sake of a concrete example that that our model society has the parameters $e = 1$, $k_1 = 1$, and $k_2 = 4$. I leave it as an exercise to the reader to use the above equations in order to determine from these the c's, v's and s's, which are:

$$\begin{array}{ccc} c_1 = \dfrac{1}{3} & v_1 = \dfrac{1}{3} & s_1 = \dfrac{1}{3} \\ c_2 = \dfrac{2}{3} & v_2 = \dfrac{1}{6} & v_3 = \dfrac{1}{6} \end{array} \tag{19}$$

Alternatively, the reader can simply confirm post-hoc that k_1, k_2 and e are all what they should be given these numbers.[7][8] Let us denote the rate of reinvestment by capitalists with the letter a, and set it to $\frac{1}{2}$ as Marx did (or at least intended to!). Finally, let us suppose initially that our system is initialized with the departmental outputs $y_1(0) = 100$ and $y_2(0) = 100$.

We've required that capitalists reinvest half of their surplus. However, we still haven't placed any constraints on yet is *where* capitalists are allowed to reinvest. A capitalist is free to invest their surplus in either department, regardless of their existing capital investments. For this initial example, assume that capitalists always choose to reinvest only in the departments that they are currently involved in. This assumption is not entirely for simplicty's sake. It amounts to the social assumption that capitalists are *lazy*. With a fully equalized rate of profit, any investment will always yield the same profit in return, so why bother doing anything new?

Now we are all set. Suppose the capitalists behave as we laid out, investing $\frac{1}{2}$ of their surplus value back into their own departments. $y_1(0) = 100$, and $y_2(0) = 100$. What would $y_1(1)$ and $y_2(1)$ be? Let's find out.

Of the 100 output value from dept. 1, the portion of this which is surplus is $s_1 y_1(0) = \frac{1}{3}(100) \approx 34$ (exact numbers will not be important for the point we are making). Capitalists from department 1 therefore want to reinvest $a = \frac{1}{2}$ of this amount back into production, i.e. 17 hours. Note that in symbols this is $a s_1 y_1(0) = 17$.

These 17 hours of value next need to be split up into value in means of production and value in labor power according to the departmental composition of capital. Since $k_1 = 1$, this means that 8.5 of those 17 hours will be going towards means of production, while the remaining 8.5 will be going towards labor power. This second 8.5 going towards labor power is special, in that it is the part which will generate surplus value for the capitalists. The amount of surplus generated is specified by the rate of exploitation e. $e = 1$ means that we have an hour of paid labor for every hour of unpaid labor, meaning that we will generate 8.5 hours of surplus labor.

[7] This is noteworthy in its own right, as it shows that these three numbers (k_1, k_2 and e) along with the initial value outputs $y_1(0)$ and $y_2(0)$ as well as the rate of capitalist reinvestment, which we will denote a, serve to completely determine and parametrize our model.

[8] If we are to see this as a model in which multiple sectors from each department have been aggregated together (or even as a simple two-commodity economy), then it should be noted that these numbers do *not* fully determine the underlying system. For example, unit values of commodities themselves can still be virtually anything even after fixing these numbers. However, these underdetermined particularities have no bearing on the dynamics of the model we are about to construct.

This gives us everything we need to determine the output from department I at the end of the next period, $y_1(1)$. In words, $y_1(1)$ will be the old output value $y_1(0)$ plus the amount of the surplus from the previous period which was reinvested plus the surplus value which is generated by that reinvestment. That is:

$$y_1(1) = 100 + 8.5 + 8.5 + 8.5 = 125 \tag{20}$$

To be clear, the first two 8.5's here are constitutive of the 17 hours of surplus reinvested, while the final 8.5 is the surplus value generated. Let's now express this calculation symbolically. We already mentioned that the surplus reinvested was $as_1y_1(0)$. The way that this amount splits between constant and variable capital in general can be found by noting that $\frac{c_1}{c_1+v_1}$ is the proportion of capital invested which is constant capital, and likewise $\frac{v_1}{c_1+v_1}$ is the proportion which is variable capital. To put it a bit more algebraically:

$$as_1y_1(0) = \left(\frac{c_1+v_1}{c_1+v_1}\right) as_1y_(0) = \frac{c_1}{c_1+v_1}as_1y_1(0) + \frac{v_1}{c_1+v_1}as_1y_1(0) \tag{21}$$

The surplus value generated is e multiplied by the amount of variable capital reinvested. Cleaning this up a bit we have

$$e\frac{v_1}{c_1+v_1}as_1y_1(0) = \frac{s_1}{\cancel{v_1}}\frac{\cancel{v_1}}{c_1+v_1}as_1y_1(0) = \frac{s_1}{c_1+v_1}as_1y_1(0) \tag{22}$$

We now have everything we need in order to express $y_1(1)$ symbolically and clean up the result:

$$\begin{aligned}
y_1(1) &= 100 + 8.5 + 8.5 + 8.5 && (23)\\
&= y_1(0) + \frac{c_1}{c_1+v_1}as_1y_1(0) + \frac{v_1}{c_1+v_1}as_1y_1(0) + \frac{s_1}{c_1+v_1}as_1y_1(0) && (24)\\
&= y_1(0) + \frac{\cancelto{1}{c_1+v_1+s_1}}{c_1+v_1}as_1y_1(0) && (25)\\
&= y_1(0) + a\cancelto{\pi_1}{\frac{s_1}{c_1+v_1}}y_1(0) && (26)\\
&= y_1(0) + a\pi_1y_1(0) && (27)\\
&= (1+a\pi_1)y_1(0) && (28)
\end{aligned}$$

Here the cancellation on line 25 is due to equation 17 and the cancellation on line 26 is due to 11. That cleaned up quite nicely! From working this out, we have quite a bit more than just y_1 at the end of the first period. By an identical sequence of steps, we would find that

$$y_1(2) = (1+a\pi_1)y_1(1) = (1+a\pi_1)^2y_1(0) \tag{29}$$

Similarly, $y_1(3) = (1+a\pi_1)^3y_1(0)$ and so on. We thus have computed the entire growth path for department I:

$$y_1(t) = (1+a\pi_1)^ty_1(0) \tag{30}$$

Everything we just did for department I also applies to department II. If we were to repeat the argument, we would end up with

$$y_2(t) = (1+a\pi_2)^ty_2(0) \tag{31}$$

We can plug in our c's, v's and s's to find $1+a\pi_1 = \frac{5}{4}$ and $1+a\pi_2 = \frac{11}{10}$. We therefore have the concrete growth paths:

$$y_1(t) = \left(\frac{5}{4}\right)^t(100) \tag{32}$$

$$y_2(t) = \left(\frac{11}{10}\right)^t(100) \tag{33}$$

Figure 4 shows the graphs of the growth paths we just derived. This is perfect exponential growth. Everything seems smooth and prosperous.

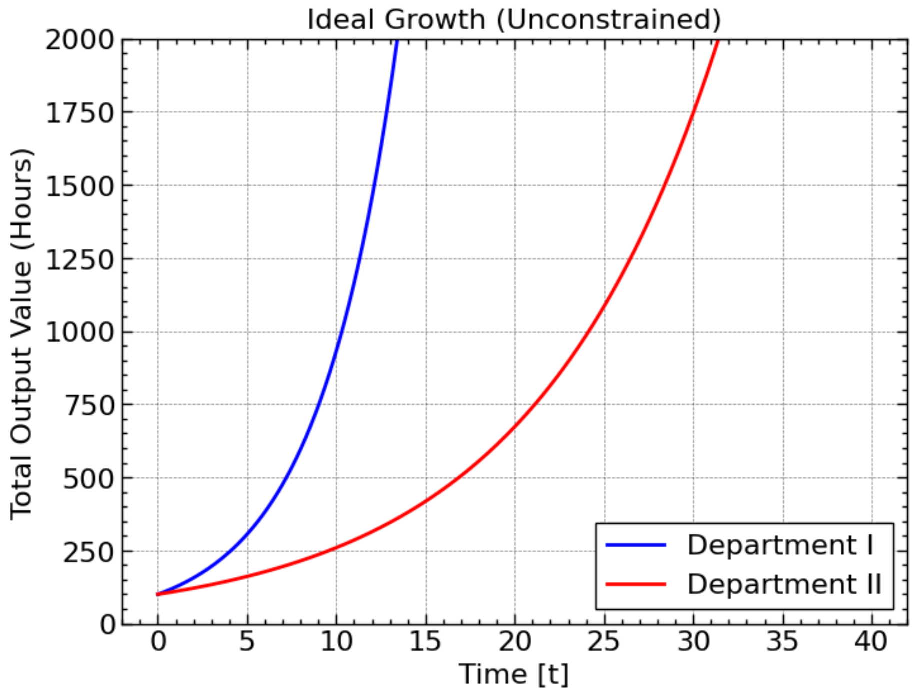

Figure 4: Equilibrium growth paths if capitalists had no physical limitations on their reinvestment.

5.3 Imposing Physical Constraints

The growth we just described assumes that the capitalist is always to pursue whatever growth they desire. This assumes two things which were not included in our initial listing(5.1):

(1) That a sufficient amount of means of production were produced inhese are the main functions to look at. the previous period

(2) That sufficient consumer goods were produced to bring in the new workers necessary for this growth.

When we remove these assumptions, a bit of labor time cost accounting will show that such growth is generally not physically possible.

5.3.1 Required Capital Goods

Consider specifically the ideal growth observed from $t = 0$ to $t = 1$. This had department I going from a value output 100 to a value output of 125. The change for department II can be computed from equation 33: it goes from 100 to 110.

To produce a value output of 125 hours, department I needs $c_1(125) = \frac{1}{3}(125) \approx 42$ hours of means of production to be available for purchase at the beginning of the period. Likewise department II needs $c_2(110) = \frac{2}{3}(110) \approx 73$ hours worth of means of production to be available. Thus, the total value of means of production which would need to be available for this kind of growth is approximately $73 + 42 = 115$. To summarize and express this symbolically (since this calculation will be important soon) we have that the required value in means of production in order to produce $y_1(t) + y_2(t)$ total value output is given by the expression

$$c_1 y_1(t) + c_2 y_2(t) \tag{34}$$

The problem is that the capitalists at time $t = 0$ *do not have the required means of production for this.* They have 100 hours worth, but they need 115. We are 15 hours short. Even leaving aside the specific machines and materials needed, these numbers simply don't work out. Before even checking to see if the necessary consumer goods exist, we can see that growth of this sort is *impossible* in our model, but let us nonetheless look at this, because here we see the issue inverted.

5.3.2 Required Wage Goods

As noted, growing the wage goods department from 100 hours at $t = 0$ to 125 hours at $t = 1$. This would require $v_1(125) = \frac{1}{3}(125) \approx 42$ worth of consumer goods to pay workers in department I, and $v_2(110) = \frac{1}{6}(110) \approx 18$ hours for workers in department II. Thus, 60 hours worth of consumer goods total are needed to pay all of the new hires necessary for the desired growth.

This isn't the end of it, though, since the capitalists also need to eat. Since we are assuming no hoarding, we know that the capitalists are going to consume whatever they don't reinvest. If dept. I capitalists are investing $as_1y_1(0)$ into production, then this means they must be consuming $(1-a)s_1y_1(0)$ hours worth of consumer goods. Likewise capitalists from dept. II need to consume $(1-a)s_2y_2(0)$ hours worth of consumer goods. In total, capitalist consumption amounts to $(\frac{1}{2})(\frac{1}{3})(100) + (\frac{1}{2})(\frac{1}{6})(100) \approx 25$ hours. To summarize in symbols, we have derived that the required value in consumer products in order to produce a total value output of $y_1(t) + y_2(t)$ is given by the expression

$$v_1y_1(t) + v_2y_2(t) + (1-a)s_1y_1(t) + (1-a)s_2y_2(t) \tag{35}$$

In summary, $60 + 25 = 85$ hours worth of labor goods are required in total for the desired growth, even though we have 100 hours available. Thus the required goods *are* available, but that proceeding along these lines (putting aside for a moment the unmet need for capital goods) would result in an *oversupply.*

5.3.3 Reflections and Observations

To summarize, growth of the sort described in section 5.2 from $t = 0$ to $t = 1$ is not possible when we impose physical constraints and require a preservation of supply-demand equilibrium. The transition of the value outputs from $(100, 100)$ to $(125, 110)$ would require more capital goods than are presently available, and would fail to consume all of the wage goods.

We argued earlier that the 'lazy' ideal growth paths are what capitalists would like to do. Within this framing, it becomes tempting to see things from the capitalists perspective. What do they see? They see that society has *too many* wage goods and *not enough* capital goods. This makes it very tempting to use words like *oversupply* and *undersupply* to reflect the that the capitalists are seeing, but doing so would be a mistake, *because our model maintains perpetual supply demand equilibrium.* What we are noting is an impulsive assessment of the capitalist class which agitates them into acting in a way which they rather wouldn't, but which nonetheless does not actually disrupt equilibrium in any real way. In order to avoid confusion, we will say that the desired growth is impossible due to an **overemphasized** wage goods department and an **underemphasized** capital goods department. To be overemphasized is not to be oversupplied, and to be underemphasized is not to be undersupplied. Rather, these words are reflective of a certain unconscious perception of the capitalist class which desires to grow in a certain way, but cannot due to either a lack of available resources, or to their unwavering dedication towards preserving stable market equilibrium.

Moving on then, is there any way to grow our economy at a fixed 50% surplus reinvestment rate while maintaining steady market equilibrium? The answer is yes, but we will need to give the capitalists more flexibility than we anticipated. Consider the underemphasized wage goods department. This department, recall, has a higher composition of capital ($k_2 = 4$), and thus any investment at all will require more capital goods (which we already have an underemphasized amount of) than an equal investment in the capital goods department. It appears therefore as if the only way to pursue growth on the order of 50% surplus reinvestment will require us allowing capitalists to *divest* their investments from the wage goods departments into the capital goods departments, to correct the 'emphasis imbalance'. Only by a *contraction* of the wage goods sectors of the economy can we hope to grow the overall size of the economy!

This is a startling realization, to say the least. The political economists of Marx's time likely would never have believed that supply-demand equilibrium could be tumultuous enough involve spontaneous contractions of entire sectors of the economy. Yet what we will show is that such contractions are *required* to have any hope of maintaining equilibrium while simultaneously growing in the way capitalists would realistically want to.

To complete the numerical example we started, it turns out that there is only a single possible pair of values for $y_1(1)$ and $y_2(1)$ which would see capitalists reinvesting the full 50% of their surplus value accumulated during the period $t = 0$:

$$y_1 : 100 \to 200 \tag{36}$$

$$y_2 : 100 \to 50 \tag{37}$$

Thus, our society would see the wage goods department contracted to half of its size, and the capital goods doubled in size. Note that this means *mass layoffs* - half the workers in department II would be let go, *even in equilibrium!* We will return to this point later, but now we should take a moment and note that this was likely where Marx began to struggle. It's easy enough to confirm (and I recommend that the reader do so) that this growth is indeed possible within the resource constraints without disrupting supply-demand equilibrium, and that such growth does in fact correspond to 50% surplus reinvestment. Actually calculating the numbers, on the other hand, is tricky. The real issue, more likely, is simply a matter of mathematical maturity. The kind of training required to navigate these considerations requires formal training for most which Marx simply did not have access to.

Nonetheless, we can start to see a glimpse of Marx's completed argument bearing out some very harsh conclusions about the nature of growth under capitalism, even in the idealized conditions of perpetual equilibrium. These were the conditions which the political economists believed would be utopian in nature, and yet we are seeing the inevitability of mass layoffs.

But those layoffs are not even the end of it, since the new numbers we are left with have the same issues in reverse. We are now faced with an overemphasized wage goods department and an underemphasized wage goods department! Thus the disruptions to the labor force will continue, with even greater intensity, from $t = 1$ to $t = 2$. More layoffs, and more chaos, with no end in sight.

5.4 General Expression of the Model

The equations 34 and 35 expressing our resource constraints fully characterise our system. We express them as a system here in order to complete our description of the model. Letting $b = (1-a)$ denote the proportion of surplus value which capitalists will consume from one period to another, we have that the growth paths $y_1(t)$ and $y_2(t)$ must satisfy the physical requirements

$$y_1(t) = c_1 y_1(t+1) + c_2 y_2(t+1) \tag{38}$$

$$y_2(t) = v_1 y_1(t+1) + v_2 y_2(t+1) + b s_1 y_1(t) + b s_2 y_2(t) \tag{39}$$

In plain English, we are saying here that the total value of the means of production available at the beginning of a production period must always be exactly equal to the value of the means of production required for the projected growth (computed in the way we discussed in section 5.3.1), and that the total value of wage goods must always be exactly equal to the total value of wage goods consumed by workers plus the value in consumer goods consumed by capitalists. This total, computed as we discussed in section 5.3.2, involves consumption by the capitalists of an amount $(1 - a)$ of their surplus value obtained from production. Thus the required reivestment rate is contained here as well, albeit implicitly. Finally note that the requirement of perpetual supply-demand equilibriumis also present here in in the equal signs. A relaxed model which merely requires that the materials be consistently available (but allowing for disruptions to supply and demand) can easily be obtained by replacing the equal signs with $\geq$ signs, thus turning our system of equations into a system of inequalities. Neither Morishima nor myself have tried to think about what changes in the analysis by relaxing this constraint, but such an investigation could be very interesting in its own right.

The process of mathematical discovery we just walked through to get to these equations is often underlooked as a source of difficulty in mathematics, but I would argue that it is the hardest part. It requires the kind of training and practice in formal reasoning which Marx never had access to, and maybe even did not exist during his time period. Not only this, writing these equations down would not have even been useful to Marx, since solving such a system, either analytically or by way of numerical approximation, would be far out of his reach.

Indeed, those who are more versed in mathematics will recognize this as a system of difference equations. To 'solve' such equations is to solve for an entire pair of functions y_1 and y_2, not just a pair of numbers. You can choose to see the solution to such systems either as a pair of infinite sequences of numbers, or alternatively as a pair of curves. One is typically not taught how to solve such systems until taking a differential equations class in college. To quote Morishima on the matter:

"The unnatural adjustment of the rate of accumulation by capitalists of department II to the exogeneously determined rate of accumulation in department I was invited by Marx merely as a *deus ex machina*. However we should not be too surprised that Marx performed so poorly in this case. Even Walras could not properly solve the simultaneous differential (or difference) equations describing the process of *tatonnement*. Remembering that Marx had begun his academic career as a philosophical student and learned mathematical economics by himself, we should be greatly impressed by his model, which may be taken as the prototype of the present-day Leontief-von Neumann models. In mathematical economics, like other sciences, the most important thing is to pose fundamental problems. Once models have been formulated, their solution may be relegated to assistants or even computers."([13]p.125)

On that last point, back in 1973 when Morishima wrote his book, we should not be surpsised that he himself only foresaw computers as a way to solve such equations. Indeed, computers can do much more than this - they can allow us to *explore* the set of solutions. The solution itself is not too difficult to find analytically for those already versed in solving systems of differential equations - that process, as well as some formal results about the set of solutions, can be found in appendix A, or in Morishima's book.

The exact mathematical form of the solution to this system is not important to us. Instead, I've included a companion 'app' which allows us to visualize and explore the solutions. Let us now turn to that, and see for ourselves what Marx's final punchlines to volume II ought to have been.

6 The Crisis of Disproportionality

A desmos 'app' made to visualize our model can be found by clicking here. Desmos is a free online browser based graphing calculator, and a very unconventional choice for this kind of application. I initially chose to use simply out of convenience, as a learning tool for myself. However, I quickly found myself adding more and more to it, until it suddenly felt like an extremely robust learning tool. I (along with one of my students, special thanks to Jao Van Schalkwyk for his efforts here) have tried multiple times to 'port' the app to a dedicated standalone graphical application, but every attempt felt inferior to the original Desmos model. Nonetheless, Jao's source code is available here, and I applaud his valient effort. Anyone who wishes to add to that project should feel free to do so and submit a pull request. I've included as a companion to this paper a brief guide on how to use the app, and would encourage readers to experiment with it and follow along as they read these next two sections.

Key parameters of the system are adjustable as sliders. These include:

- The initial value outputs of the two departments (denoted y_{1i} and y_{2i} respectively)
- The overall rate of exploitation e
- The rate of surplus reinvestment a
- The compositions of capital for the two departments, k_1 and k_2 respectively

hese are the main functions to look at. These six numbers fully characterize the system and produce different equilibrium growth paths. Pictured in figure 38 are the curves $y_1(t)$ and $y_2(t)$ (found under the) which solve the difference equations for our numerical example in section 5 ($y_{1i} = y_{2i} = 100, k_1 = 1, k_2 = 4, e = 1, a = \frac{1}{2}$). These are the only curves which should be visible initially, although there are many more functions which can be toggled.

6.1 Warranted Growth

Figure 5 shows the solution to the graph of the functions $y_1(t)$ and $y_2(t)$ which solve our particular numberical example ($y_{1i} = y_{2i} = 100, k_1 = 1, k_2 = 4, e = 1, a = \frac{1}{2}$). Morishima calls these the *warranted growth paths*

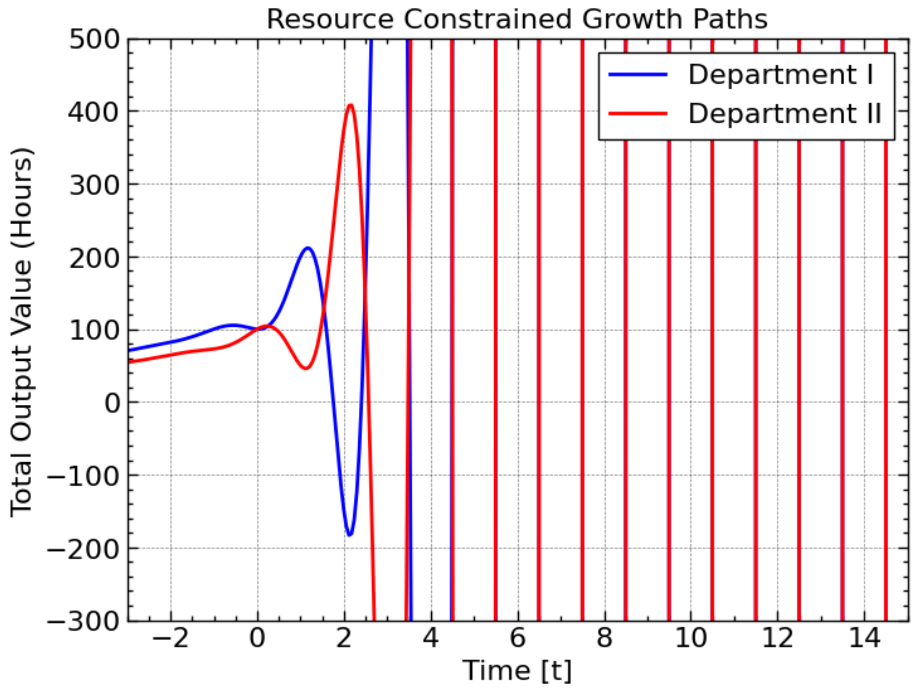

Figure 5: Growth paths for our numerical example

of the system, and we will discuss shortly how they should be interpreted. These should also be what the reader is greeted with if they visit the app themselves in Desmos. The formulas defining the graphs are found under the "Growth paths of the system" grouping. There are many other togglable functions which can be viewed here, but in general, as a rule of thumb, purple curves will be related to department I, while red curves will be related to department II.

What do these warranted growth paths represent? They represent the total value outputs from the two departments which *must* be followed in order to allow the capitalist class as a whole to reinvest the proportion of their surplus value a while maintaining perpetual supply demand equilibrium.[9] It really cannot be stressed enough that these curves define how the market evolves in conditions of perpetual supply-demand equilibrium. The explosive fluctuations we see are not the result of supply-demand fluctuations. A capitalist society developing along these curves *will never notice any price fluctuations as a result of supply falling below demand or vice versa.* Thus, any attempt to understand why the capitalist class based on supply-demand fluctuations is *faulty.*

However, what we see nonetheless is wild swings of favorability between the two departments. As we discussed, at period 0 of our particular example, the capitalists perceive an over*emphasized* (not oversupplied!) wage goods department and an under*emphasized* capital goods department. To the extent that this can be seen as a 'real' perception from actors living in our model economy, it is felt *not* through the price of wage goods falling, but rather, perhaps, through the *agitation* of capitalists having their reinvestment ambitions disrupted. This can be seen as playing out in a variety of ways. For example, we can imagine capitalists in department I first reinvesting their surplus first, as they want. In this situation, capitalists in department II would find that the means of production they need not only to reinvest their surplus but also to simply restart their production at the same scale as the previous period unavailable. 'Humble servants of the market' as they are, and thus unwilling to disrupt the delicate price equilibrium by outbidding by paying extra for the product they need, decide instead to reinvest where they can based on the available resources - namely in department I. What we are left with at the end of the production period is an even more unbalanced situation. The wage goods sectors are now massively underemphasized and the capital

[9] We are choosing to see these graphs as continuous curves for the sake of intuition and visualization, but it should be noted that the functions y_1 and y_2 are really discrete functions and are only formally defined on integer inputs.

goods overemphasized. Supply and demand remain in equilibrium.

Is this story I just told realistic at all? Of course not. This is where it becomes important to recognize that our intent was never to create a predictive model of the economy. Our concern is not with *how* or *why* the capitalist class keeps society on these warranted growth paths. Rather, it is to understand what is *socially necessary* for the reproduction and expansion of a capitalist system in perpetual supply-demand equilibrium. What matters is that if the capitalist class wants a such a society, then they *must* stay on these growth paths, one way or another.[10] So let's assume they somehow manage to, and see what happens next.

What happens next, between periods 1 and 2, is crucial. Here we see that the required values of $y_1(2)$ and $y_2(t)$ are approximately -352 and 550, respectively. A negative value output is impossible, so at this point, the number of options that the capitalists have for their desired growth truly goes to 0 - it is at this point completely impossible. But that won't stop them from trying. If we are to think about the reinvestment scramble as a continuous process from time $t = 1$ to $t = 2$, what we would eventually be faced with is the *complete liquidation of the capital goods department.*

At this point, the inhabitants of our model society would be forced to admit it was facing some kind of crisis. What is the nature of this crisis?

6.2 The Crisis of Disproportionality

The Russian Marxist Tugan-Baranowsky was the first to derive a kind of disproportionality theory of crisis from an inspection of Marx's reproduction schema, spawning a critique that became known as the *disproportionality crisis theory.* Through a thorough inspection of the schema (and indeed an expansion of it into a three department model to allow for luxury goods) which utilized the same peculiar investment scheme Marx used, Tugan-Baranowsky concluded the only thing one can about the conditions - that there is very little reason to expect capitalists to be able to satisfy them. He writes: "capitalism possesses no organization capable of guaranteeing this proportionality. Industrial crises derive from this". His conclusions went quite a bit farther than this - he then claimed that these crises of disproportionality were the *only* explanation for crises.

Tugan-Baranowsky continued to give a detailed account of the business cycle - indeed he can be called one of the early pioneers of trade cycle analysis. However, the connection between his extension of Marx's reproduction schema and his business cylce theory is quite tenuous and often disputed. writes: "We conclude by saying that the linkage between the theory of disproportionality and that of the business cycle proves to be problematic: if the former may explain the final breakdown of the economic system, the latter requires further investigation - especially with respect to the determinants of investment and money demand - and these unexplained elements can be seen as the legacy of Tugan for the next generation of economists". On closer inspection, Tugan's analysis of the business cycle is at best a theoretical illustration of the more standard Marxist theory of overproduction, which we will talk about shortly. In particular, it is felt in the same way - through a stoppage of the general flow of capital circuits as consumers of products fail, for whatever reason, to materialize. While it may have been inspired by Marx's reproduction schema, it has nothing directly to do with it.

I am going to call the event we are witnessing here, in which one department suddenly and rapidly becomes entirely liquidated within the optimal course of perfect market equilibrium growth, a crisis of disproportionality. There is simply nothing else to call it. I say *a* crisis of disproportionality, not *the*, because this is by no means the only way for a disproportionality crisis can occur. However, I believe that this particular emergence of the crisis of disproportionality has special significance and major implications for Marxist crisis theory, and it is not something which I've seen acknowledged anywhere else.

Unlike Tugan-Baranowsky's so-called crisis of disproportionality, what we are witnessing in ours can in no way be confused with the typical Marxist crisis of overproduction or underconsumption. This is for three reasons. First, because *at no point does any overproduction or underproduction actually occur.* Second, because it arises from a completely different (and more foundational) contradiction of the capitalist system than the crisis of overproduction. Thirdly, because I believe that the effects and implications of this crisis are very different from those of the crisis of overproduction. I will elaborate on the first two of these reasons here. The third I will save for a discussion in section ???.

[10] This is by the uniqueness of the solutions up to our parameters. See A for proof of this.

The classical Marxist theory of crisis stems from the contradiction between increasing exploitation of workers and increasing productivity in the methods of production. To quote Marx himself, he says this within a footnote of volume II:

Contradiction in the capitalist mode of production: the labourers as buyers of commodities are important for the market. But as sellers of their own commodity labour-power capitalist society tends to keep them down to the minimum price. -Further contradiction: the periods in which capitalist production exerts all its forces regularly turn out to be periods of over-production, because production potentials can never be utilised to such an extent that more value may not only be produced but also realised; but the sale of commodities, the realisation of commodity-capital and thus of surplus-value, is limited, not by the consumer requirements of society in general, but by the consumer requirements of a society in which the vast majority are always poor and must always remain poor. However, this pertains to the next part."([9]p.???)

By 'next part' here, Marx is referring to part III as a whole in which he inspects the three major circulations of capital one by one. This is important to note because the contradiction, as he puts it, entirely pertains to circulation β, and *not* circulation γ, which is where we discovered the crisis of disproportionality. Even more importantly, Marx is very explicit about the contradiction here, that between the development of the productive forces, on the one hand, and the restricted consumption of the impoverished masses on the other.

Observe now that the crisis of disproportionality we is not brought about by a lack of required consumption spending by the working class, nor by an overproduction of goods for that class. The productive forces are not developing at all - technical compositions of capital are frozen as an assumption of the model. All produced goods are consumed, always. If we call the event we are seeing a crisis, then there is simply no way to view it as being due to the contradiction driving the classical Marxist crisis of overproduction. What contradiction then is our crisis a manifestation of? can we say this is a manifestation of then? The answer is the fundamental contradiction found in commodities themselves, which Marx details in the opening chapters of Volume I: that between *use-value* and *value*.

Capitalism is a mode of production. Its purpose is to produce use-values for the humans living within it - useful things which we need to survive and thrive. The distinction between departments I and II - between means of production and consumer products - is one of the most basic distinctions we can make with respect to describing distinct categories of use-values. Means of production goods are products whose use-values lie only in their function within production itself, while wage goods are the use-value lies outside of production, as end-products. The contradiction is that in capitalism, capitalists produce use-values, but the specific use-values they are producing isn't of any concern to them. The capitalist is motivated to produce not for use, but rather for exchange. Rather than use, the capitalist is entire concerned not with the use-values they are producing, but with their *values* - particularly the surplus value component of those values which correspond to them (indirectly) in the form of profit.

Here, to put it as Marx would, what we see in our crisis of disproportionality is precisely the contradiction between use-value and value "sharpening to the point of an absolute contradiction". The capitalist class, blind to all use-values and entirely pursuant of exchange-value, makes no distinction between consumer goods and means of production. The capitalists in our system do not care whether they are making purses or the leather used for making purses. Both are just numbers to them, equal up to some proportional relationship (x square inches of leather = 1 purse). What our model shows is the crisis created when capitalists, in their exclusive pursuit of exchange-value, catastrophically fail to create use-values in the proportions which society requires for it's own social reproduction. Capitalists, motivated by the need for constant multiplicative growth, not in the mass of use-values producing but instead by the total combined values of those commodities, are seen not to produce the means of production needed to renew their production on the next cycle, nor the consumer goods needed by the workers. Rather, they are seen blindly invest their resources wherever they can, and even ceasing existing production in order to force the growth they want. The correspondingly wild swings of favorability between departments gives us what we are calling a crisis of disproportionality.

In volume III of capital, Marx writes: "a crisis could... be explained as the result of a disproportion of production in various branches of the economy, and as a result of a disproportion between the consumption of the capitalists and their accumulation. But as matters stand, the replacement of the capital invested in production depends largely upon the consuming power of the non-producing classes, while the consuming power of the workers is limited partly by the law of wages, partly by the fact that they are used only as long as they can profitably be employed by the capitalist class. *The ultimate reason for all real crises* *always remains the poverty of and restricted consumption of the masses as opposed to the drive of capitalist production to develop the productive forces.*"([10]p.484)

This quote is often wielded by Marxists as a bludgeon against those who would challenge the supremacy of the crisis of overproduction. I empathize here with Marx's desire to centralize this particular contradiction as being of prime importance to the eventual downfall of capitalism, but I believe that in making the claim that it is the *ultimate reason for all real crises* he fails to give his own theory the proper credit. All throughout the volumes of Capital, Marx is identifying contradictions of the system which will - must - drive its historical development, and which lead to crisis under the failure to do so. Not all of these contradictions follow from a single fundamental source, nor should they. For example, we already recounted in our inspection of the intra department I circulation (circulation α) the contradictions of fixed capital and hoarding which could lead to a crisis of the supply chain.

Perhaps it should be noted that the word 'crisis' itself has a bit of a dual meaning among Marxist thought. At times, it is used to refer to the periodic crises which occur every thirty or so years (e.g. the Great Depression, the profitability crisis of the 70s, the 2008 financial crisis). At other times, it is used to refer to the messianic crisis of prophesy which has not happened yet - the one which will lead to the true overthrow of capitalism. My personal belief is that Marx's 'ultimate cause for all real crises' here refers to the latter of these uses of the word and not the former. In this sense, I can at least morally agree with the claim - failure for real demand to keep up with increasing productivity in the face of stagnant wages remains by far the largest overall existential threat to capitalism, and has been since the crisis of the 70s. I still disagree that it is the only existential threat - the crisis which ends capitalism will always fundamentally be historically contingent, but I can at least see and respect where his claim is coming from.

However, we must observe that the logical content of what Marx is saying here is very flawed. The only moment where the workers' purchasing ability can disrupt the replacement of capital invested in production is in the transformation of newly produced commodities into money: $C \to M$. However, this is *not by any means* the only place where the replacement can be interrupted, and we see that very clearly in our model. After all, our model is one of continual equilibrium; all finished goods are always sold. Yet, within the context of this most-severe disproportionality - the total liquidation of an entire department - capitalists would indeed find the replacement of their capital investment halted, either by an inability to turn their capital in the form of money into means of production and/or labor power. In other words, the disruption we are witnessing is not felt in the moment $C \to M$, but rather the subsequent $M \to C$. Marx's mistake here is a classic case of value-form fixation. Indeed, the crisis is completely invisible from the perspective of capital circuits *when the focus is exclusively on money* - it only reveals itself in the 'real' world of use-values. This is what makes the phenomena we are witnessing so interesting, and so important. It shines a light on a moment in which those two worlds fail to coincide. In the face of our results, we must therefore assume that Marx would have revised this statement if he had found his way to them himself.

To summarize, the crisis of disproportionality we are witnessing in this model has nothing to do with what Tugan-Baranowsky identified, because it is not felt through the price signal - there is no undersupply or oversupply of anything. This crisis occurs independendly of price signals or the value-form value entirely, and only can be perceived in the world of use-values - when the capitalist goes to market to exchange their renewed money-capital for useful things they need for production. Furthermore, it is distinct from the classical Marxist crisis of overproduction, since it arises from a different contradiction inherent to capitalism, present within a different overall circulation from that where the overproduction crisis can be witnessed, and more importantly because it halts capital circulation at a moment which has nothing to do with workers' consumption. It remains to convince the reader of the importance and significance of this distinction. In order to do that, we must first turn to an important question: can these crises of disproportionality be avoided?

6.3 The Statistical Impossibility of Balanced Growth

Can the capitalists avoid the crisis of disproportionality? The crisis is guaranteed within our particular numerical example (in fact, it is arguably already underway at time $t = 0$). However, these growth curves are determined by a set of parameters: the rate of exploitation e, the rate of surplus reinvestment a, the compositions of capital k_1 and k_2, and finally the initial value outputs y_{1i} and y_{2i}. Every different choice of six numbers to assign to these parameters yields a different pair of growth curves. Is there a choice which

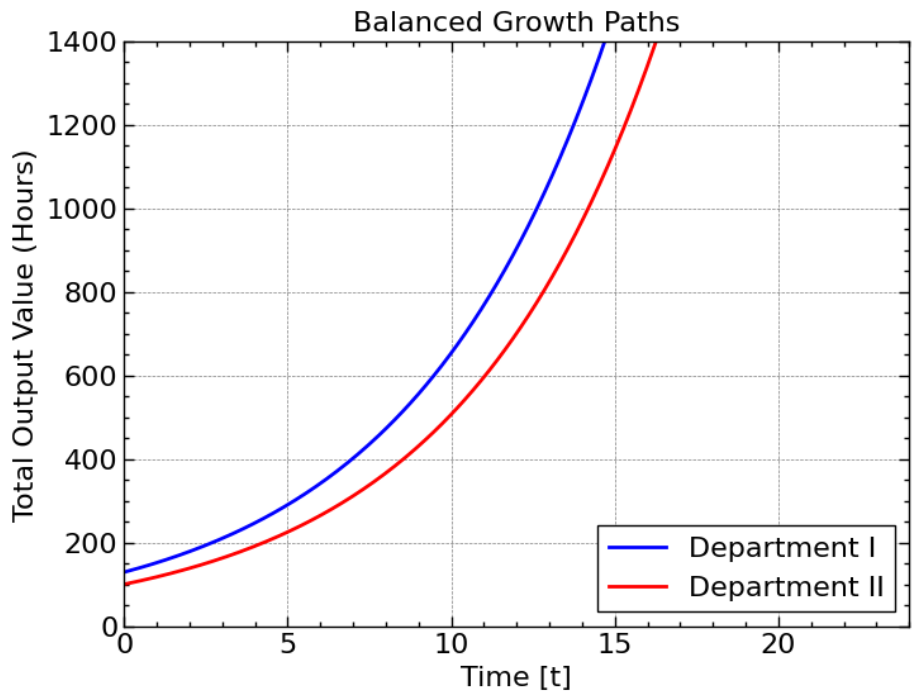

Figure 6: An example balanced growth path. Here all parameters are the same as our numerical example from section 5, except with $y_2(0)$ changed to accomadate $y_1(0)$.

would be desirable to the capitalists?

Before proceeding, we should try to express a little more clearly what kinds of growth paths we are looking for in order to prevent the crisis of disproportionality from occuring. In principle, we should be looking for growth paths which look something like the ideal growth curves that we observed earlier in section 5.2. However, for the sake of our inquiry we can set the bar quite a bit lower. Can we even find growth curves in our model such that *both departments stay consistently positive* in their output?

6.3.1 Effects of the Rate of Exploitation and the Rate of Reinvestment

The reader is invited to play around with the sliders for a and e and see for themselves the effect that it has. They would see that while these parameters do have significant effects on the shape of the growth paths, they do not do so in any way which prevents one of the departmental outputs from eventually going negative. Indeed, they will see that these parameters also do nothing at all to effect the *timetable of the crisis*. Disruptive oscillations will begin to occur at the same time regardless of how these variables are adjusted. Thus the capitalist class has no hope of avoiding the crisis of disproportionality through adjusting their treatment of workers or through an adjustment of their reinvestment rate.

6.3.2 Effects of the Initial Value Outputs For the Departments

Unlike a and e, careful choice of y_{1i} and y_{2i} do offer the vague promise of a way out for the capitalists. Morishima shows (as do I appendix A for those interested), for a particular initial choice of y_{1i} there is a single, very specific choice of y_{2i} which produces what we will call (as Morishima does) *balanced growth.*[11] The same is true in reverse: any choice of y_{2i} yields a single value of y_{1i} which produces this balanced growth.[12] An example of such balanced growth path is shown in figure 6.

[11] The choice of words here is because one can observe that the ratio of total value output of the departments against each other will be perpetually constant (e.g. for some constant $\alpha \in \mathbb{R}^+$, $\frac{y_1(t)}{y_2(t)} = \alpha$ for all t.

[12] Thus the set of all pairs (y_{1i}, y_{2i}) which produce balanced growth constitutes a *ray* in $\mathbb{R}^2$.

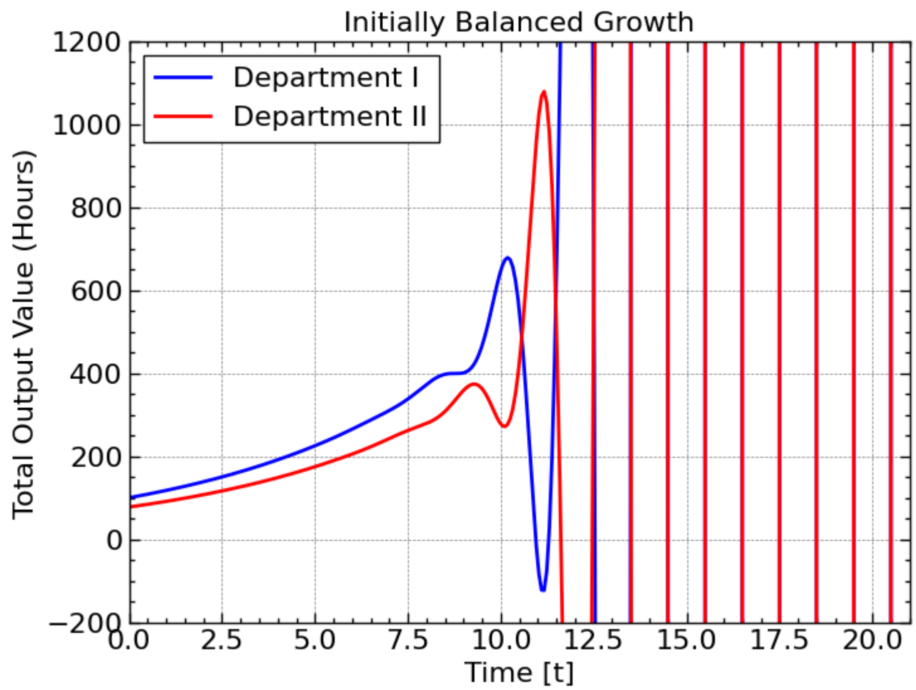

Figure 7: As the initial values of the departments approach what is required for balanced growth, the crisis of disproportionality is delayed, but remains inevitable.

In the Desmos app, underneath the sliders for y_{1i} and y_{2i}, two numbers are displayed named b_{g1} and b_{g2}. With y_{2i} fixed, b_{g1} is the number that y_{1i} needs to be set to in order to achieve balanced growth. b_{g2}, similarly, is what y_{2i} would need to be set to for a fixed choice of y_{1i}. The reader is encouraged to try setting one of the departments initial value outputs according to these numbers to see the effect that this has. The output graphs adjust in real time with every digit typed. As one types in more significant figures, they will see the crisis deferred to later and later production periods in the future. Figure 7 shows our numerical example from section 5, except with $y_2(0)$ set to 77.532 instead of 100, which is approximate up to the one ten thousandths of the magic number b_{g2}. Observe that even with this level of precision, we have only deferred the crisis by about 10 production periods.[13]

Thus, balanced growth is achievable within our model, after all. Unfortunately though for the capitalists and the utopian political economists, this is an empty promise. Even if the capitalists were aware of this necessary ratio of initial conditions and were able to somehow coordinate to 'kick off' their society based on these initial outputs, in practice the crisis would still be inevitable. This is because in the real world, we can never hope to achieve perfect accuracy. For the same reason that it is statistically impossible to find someone on the street who is *exactly* 6 feet tall, it will likewise be statistically impossible for the capitalists to tune these numbers to an *exact* pair of real numbers which brings about balanced growth, let alone manage to stay on those curves afterwards.

To put it more mathematically, the parameter space of initial conditions constitutes a two dimensional plane (the positive quadrant in $\mathbb{R}^2$), while the subset of initial conditions producing balanced growth is a one dimensional line living in this plane. If we try placing a pin on this line by hand, we will, with statistical certainty, always fail. Try as we might, if we were to zoom in, then we would always find that we have missed the line by some small amount, because the 1D line constitutes a measure 0 subset of this 2D space. Moreover, *any deviation at all* from the required b_{g1} will guarantee explosive divergence after a small number of production cycles.

[13] Even if the reader types in the entire displayed number for b_{g1} for y_{1i}, they will still see an eventual crisis.This is because Desmos is performing some truncation with the value it is displaying. If the reader wishes to see the balanced growth path in its entirety, they need to type in the symbolic expression, e.g. instead of setting y_{1i} to the number displayed, simply set it equal to b_{g1}.

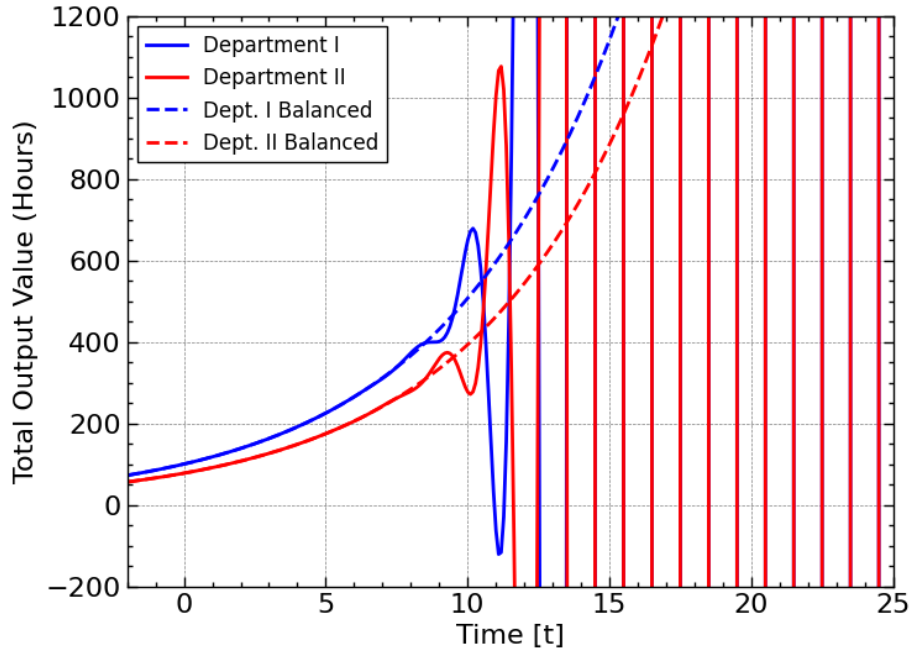

Figure 8: As the initial values of the departments approach what is required for balanced growth, the crisis of disproportionality is delayed, but remains inevitable.

Despite being impossibly rare to fully obtain in a practical sense, these blanaced growth paths are theoretically important as they centralize the dynamics of our system. To the extent that the capitalist class can approximate an initial condition (y_{1i}, y_{2i}) corresponding to balanced growth, they will find themselves approximating a balanced growth path. In fact, *all* growth paths of the system initially follow a balanced growth path until eventually exploding off of it.

In the Desmos app, under 'growth paths of the system' tab, the reader can observe this for themself by toggling the curves denoted z_{g1} and z_{g2}, and located directly underneath the y_1 and y_2 curves in the 'growth paths of the system' grouping. These dashed curves represent the balanced growth which the actual warranted growth paths are approximating before eventually exploding off from them (see figure 8).[14] Thus all growth 'walks in she shadow' of a balanced growth path. In dynamic systems terms, we would refer to these as *unstable fixed points* of the system.

6.3.3 Effects of the Composition of Capitals for the Departments

Turning to our final two parameters, k_1 and k_2 have a massive impact on the qualitative trajectory of the system. This is because k_1 and k_2 represent the degree of dependence that the two departments have *on each other*. In fact, the behavior we are witnessing is best understood causally in terms of the *difference* between k_1 and k_2.

Recall that when a capitalist goes to market to purchase capital for their production process, they must purchase means of production and labor power in a fixed proportion which is given by the current standard methods of production.[15] The ratio of how much value in means of production they must purchase vs how much value in labor power for every dollar spent is precisely the composition of capital. However, in the context of the departmental model Marx is looking at, these numbers take on additional meaning, since they directly represent the interdepartmental spending practices. Value spent on means of production *is*

[14] I call these in the app 'almost balanced', since they are not exactly characteristic of any average growth path, but are rather the 'average' of the balanced growth path obtained from holding y_{1i} constant and setting $y_{2i} = b_{g2}$ and the balanced growth path obtained from holding y_{2i} constant and setting $y_{1i} = b_{g1}$. This was chosen so as not to 'bias' the path towards either department, although doing it this way also looks very similar.

[15] It is also determined by the current value of labor power, but we are assuming this to be fixed within our model.

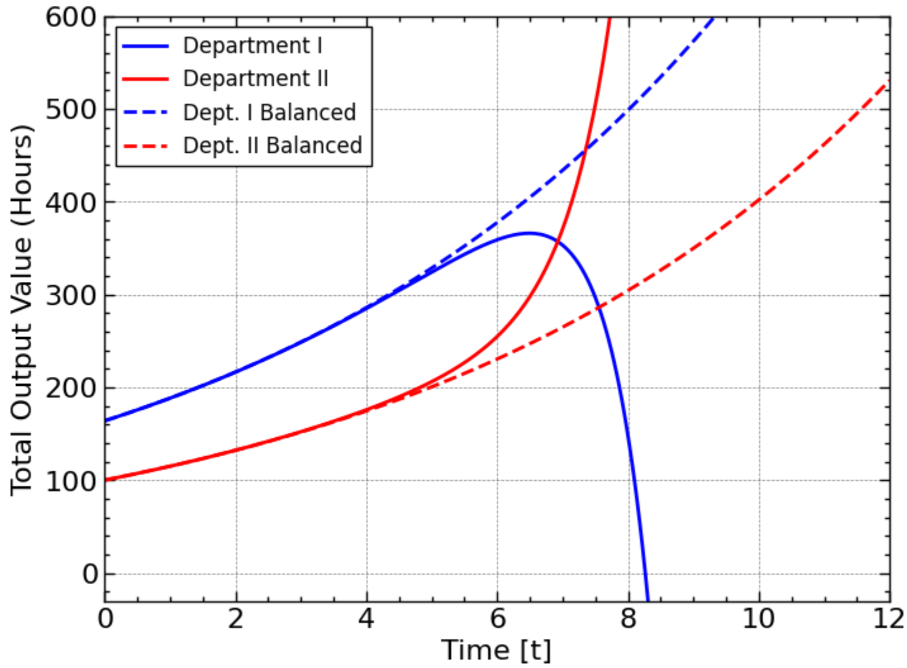

Figure 9: When department I has the larger composition of capital, we get divergence from the balanced growth paths, rather than oscillation about them. (Here, $y_{1i} = 163.8, y_{2i} = 100, e = 1, a = 0.5, k_1 = 4, k_2 = 1$.)

value spent on department I. Value spent on labor power *is* value spent on department II, though this is less direct. [16]. Thus the overall compositions of capital k_1 and k_2 define the *dependency structure of these two departments on each other.* It is the mismatch between them which fundamentally creates the crises we are witnessing, and how exactly they compare determines the qualitative nature of how the crisis unfolds.

In our particular example with $k_1 < k_2$, we have that the capital goods department is more dependent on wage goods than the wage goods department is on capital goods. In other words, $k_1 < k_2$ represents an *inward facing dependency* between the two departments: both departments are *overly reliant* on one another. This is fundamentally why we end up with the oscillations that we are observing - each oscillation essentially represents another failed attempt to overcompensate in the face of this imbalance. The wage goods department can't just keep reinvesting in itself because that would require too many capital goods, and the capital goods department faces the same problem in reverse. The inward facing dependency between the departments makes such a practice unsustainable. The only way to consistently grow in terms of total value output is to allow for the currently overemphasized industries to contract, but again the inward facing dependency structure combined with our reinvestment constraints guarantees that this course correction measure will be overcompensating, requiring an even greater contraction of the currently underemphasized department in the next production cycle. The overall result is *unstable oscillation.*

If the inward facing dependency just described and characterized by $k_1 < k_2$ is the root cause of the explosive oscillations we see in the model, then it would have to follow that we see something qualitatively different if the situation is reversed. Indeed, this is the case, and I invite the reader to exchange the current values for k_1 and k_2 and see this effect for themselves. Figure 9 shows an example of what this looks like.

It turns out that avoiding oscillations doesn't amount to avoiding the crisis of disproportionality. Instead of oscillation, we are greeted by a purely *divergent* trajectory, in which one department grows without bound and the other fades into oblivion. This is to be expected, since the dependency structure of $k_2 > k_1$ is one of outward facing dependency: *both departments are overly reliant on themselves.* In a situation such as this, the department which is initially underemphasized becomes a sinking ship. All capitalists gradually divest

[16]This money goes to department II only after the workers spend their wages on consumer goods, but the entirety of the money spend on labor goes to department II eventually

from it and move to the only game in town until the crisis hits.

The final possible relationship to consider between k_1 and k_2 is when the two compositions of capital are equal. If the reader attempts to set $k_1 = k_2$ in the Desmos model, then the model will break. This is simply because setting k_1 and k_2 equal undermines a fundamental assumption about our system - that the two department are distinct. This theoretical model has no criteria for distinguishing the two departments from each other if the compositions of capital are the same. Morishima himself shows in chapter 8 that if the compositions of capital are equal, then the departments can be validly aggregated together and seen as a single department.([13] p.93) In this case, the math which brought us here needs to be revised, but it can easily be seen that as long as capitalists invest in both department in a certain fixed proportion, balanced growth is very possible and not particularly difficult.

However, this should not provide any solace to the capitalist class or the political economists, for an identical reason to what we saw before when inspecting the initial value outputs. It will never be the case that k_1 is *exactly equal* to k_2.

To summarize, generally prosperous and non-disruptive growth *is* achievable within our system. However, the balanced growth paths are *unstable*. All conditions which allow for balanced growth, or for that matter any growth which simply prevents one of the departments from eventually careening out of existence, are statistically impossible to reach in practice. The crisis of disproportionality is therefore very different from other crises which Marx discusses in the volumes of Capital.

6.4 Implications for a Marxian Theory of Crisis

6.4.1 The Inevitability of Crisis

As we mentioned, the crisis of overproduction, as well as other crisis directly discussed or implied by Marx, arise out of dialectical tensions within the capitalist system, but by nature of the dialectical approach taken by Marx rarely arise as inevitabilities of the system functioning ideally and as intended, e.g. as *social necessities*.

Take for example the unemployment crisis which Marx *implies* (he does not ever call it a crisis by name) within his discussion of the general law of capital accumulation in volume I. Here, Marx is identifying the hidden mechanism by which capital regulates the size of the reserve army of labor. He notes that increases in labor productivity brought about by capitalist competition have the inadvertant side effect of freeing up workers and inflating the size of the industrial reserve army. This inflation serves to counteract the general deflation which must be occurring as capitalists reinvest their surplus value and scaling up their production. These two mechanisms look as if they can correct for one another in keeping the reserve army at a nominal size which stabilizes the value of labor power and consequently wages. There is an interesting tension here. What Marx is able to conclude is that the *absolute* size of the reserve army must grow as capital accumulates, but changes to the *relative* size, which would be required to produce a true crisis for the system, are merely left as likely possibilities.

There *is* a dialectical contradiction which implies the high likelihood of this happening. On the one hand, individual capitalists are only hiring workers for the scaling up of their own local operations, while on the other hand a single individual capitalist, by their pioneering a new labor saving technology, will their own freeing up an entire percentage of the laboring population through the widespread adoption of the new production technique. Thus the layoff mechanism seems to run a very high risk of *over-inflating* the reserve army, creating an unemployment crisis.

The contradictions, and potential crises arising from them, that Marx identifies throughout the volumes of Capital, represent tensions which capitalism will need to work through over the course of it's historical development. However, it remains the case that these are *not* socially necessary - they are tendencies, *not* inevitabilities.

The crisis of disproportionality being witnessed is different, because it presents itself not as a consequence of the capitalist system failing to resolve it's internal contradictions, but rather as a consequence of the capitalist system *working perfectly as envisioned by the classical political economists*. It remains a tendency, like the other kinds of crisis, but for a completeley different reason. Where the crisis of overproduction presents itself a tendency due to the fact that it can, theoretically, be avoided in a system which functions perfectly, the crisis of disproportionality is a tendency because the system cannot be expected to function perfectly. It thus completes the critique of capitalism: whether the system successfully works through its internal contradictions or not - whether it functions perfectly or fails - crisis is inevitable.

6.4.2 The Social Necessity of Crisis

We can go a step further, and note that not only are crises revealed by this completion of the argument of volume II inevitable, they are also *socially necessary*. Whether the disproportionalities between departments cause the crisis directly or are caused by a crisis of another kind, our model reveals that market forces cannot on their own resolve disproportionalities of the sort we are identifying. If this is the case, then the only means that capitalism has of resolving these disproportionalities is through a crisis! Times of crises afford capitalists the special circumstances that they need in order to abandon all conventional rules and adapt to its conditions and avoid annihilation. Excuses can, will, and must during these times be made to correct the disproportionalities created by the market, through the destruction of a large portion of the goods produced by the currently overemphasized department.

Thus the inevitable and recurrent crises of capitalism are revealed to be a feature, not a bug. They serve an essential function which capitalism has no other means to address, and give it a special ability to adaptat and change as needed which previous modes of production were lacking. Marx fell just barely short of arriving at these conclusions himself in volume II, and we will return to summarize how this plays into both of his dual intensions for the volumes of Capital in the conclusion.

Before that, however, we need to next observe how these crises of disproportionality these crises affect the people involved - namely the working class.

6.5 The Working Class and the Demand for Labor

We have seen that the growth of a capitalist society in equilibrium following from fixed proportional reinvestment of surplus requires, paradoxically, that one department must contract in size, with capitalists divesting and shifting their investment to another department, in order to cope with resource constraints arising from differences in the departmental compositions of capital. This must correspond with workers being let go *en masse* from the shrinking department. This on it's own is incredibly disruptive to the lives of those workers, who must find new jobs elsewhere. Now, however, we must ask an even more concerning question: will these workers who were let go be able find new jobs at all?

Suppose that $k_1 < k_2$, so that department I is more labor intensive than department II, and suppose that between production periods, department II experiences a contraction. In this case, there is no issue. The capital sloshing from department II into department I requires *more* workers, not less. All laid off workers will be able to find new jobs in department I, and even beyond that, new workers will need to be brought into the production process, amounting to a reduction in the size of the industrial reserve army.

However, if department I experiences the contraction, then this spells trouble for the working class. Capital is now moving from more labor intensive to more capital intensive industries which require less workers per dollar of investment. This means that there is *no guarantee that the laid off workers will find employment elsewhere in the economy*, and in fact it must be expected that they will not. This is especially bad given our assumptions that workers are not saving. Workers who do not find jobs will go hungry.

This demands a more thorough investigation. For this we will track a new quantity relating to the system, the *demand for labor*. Informally, the demand for labor at time t, which we will denote $D(t)$ relative change in the size of the workforce between periods t and $t+1$. It is not difficult to derive an expression for this quantity in terms of what we have already. First, note that

$$(s_1 + v_1)y_1(t) + (s_2 + v_2)y_2(t) \tag{40}$$

represents the part of the total value output available at the beginning period t which exists as living labor added between periods $t-1$ and t. For simplicity, we will write $l_1 = v_1 + s_1$ (e.g. the overall living labor coefficient):

$$(s_1 + v_1)y_1(t) + (s_2 + v_2)y_2(t) = l_1 y_1(t) + l_2 y_2(t) \tag{41}$$

Thus, this is also the *total number of labor hours done by workers* during this period. Likewise, $l_1 y_1(t+1) + l_2 y_2(t+1)$ represents the living labor done by workers between periods t and $t+1$. The difference, therefore, between these quantities:

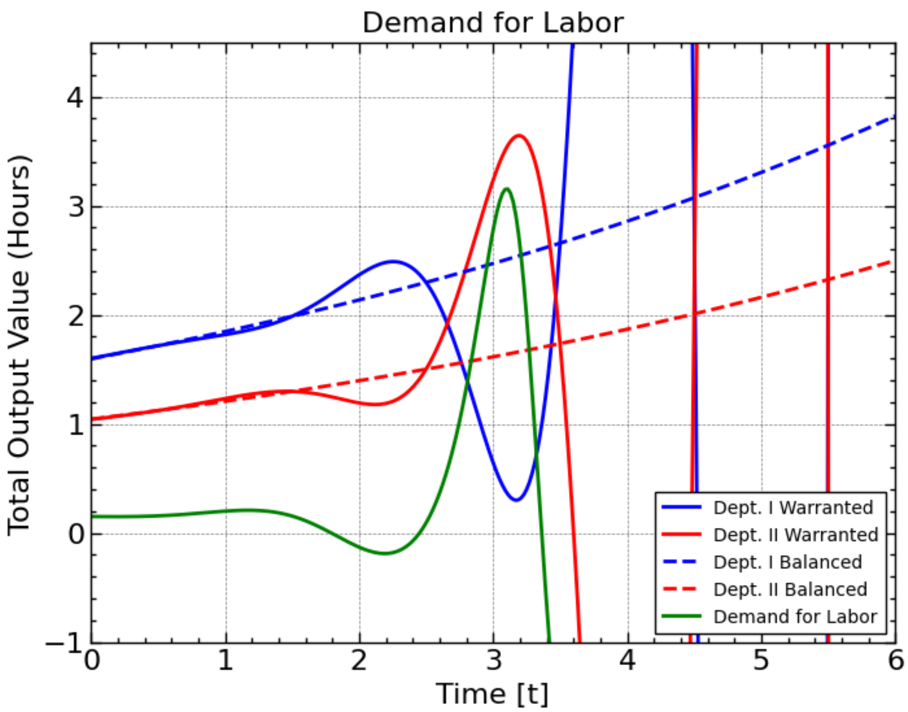

Figure 10: Plotting the demand for labor along with the value outputs from the departments. Parameters are $y_{1i} = 1.6$, $y_{2i} = 1.04, e = 1, a = 0.5, k_1 = 1.6$, and $k_2 = 3.6$

$$l_1y_1(t+1) - l_1y_1(t) + l_2y_2(t) - l_2y_2(t) = l_1(y_1(t+1) - y_1(t)) + l_2(y_2(t+1) - y_2(t)) \tag{42}$$
$$= l_1\Delta y_1(t) + l_2\Delta y_2(t) \tag{43}$$

(where $\Delta y_1(t) = y_1(t+1) - y_1(t)$ and $\Delta y_2(t) = y_2(t+1) - y_2(t)$) represents precisely how much more or less labor was done between periods t and $t+1$ than what was done during periods $t-1$ and t. Dividing this number by the total living labor done between periods $t-1$ and t gives us the relative change in labor done between periods. We thus define the demand for labor $D(t)$ by the equation

$$D(t) = \frac{l_2\Delta y_1(t) + l_2\Delta y_2(t)}{l_1y_1(t) + l_2y_2(t)} \tag{44}$$

What we wanted to track was specifically the relative change in the size of the labor force, not the relative change in the number of labor hours worked, but these are one in the same, assuming that the working day is a fixed length. If it is T hours long, then the total number of laborers employed between periods $t-1$ and t is $(l_1y_1(t) + l_2y_2(t))/T$, and the total change in number of laborers is $(l_2\Delta y_1(t) + l_2\Delta y_2(t))/T$. The relative change in the laboring population then simply has these T's cancelling out.

Thus if the $D(t) = \frac{1}{2}$, then this would mean that the total labor which will be done during the current period is 50% larger than the total labor which was done during the previous period. This amounts to a 150% increase in the size of the work force. Likewise, if $D(t) = -\frac{1}{2}$, then this amounts to a 50% reduction in the size of the work force,meaning that 50% of workers will be laid off and not find re-employment during the current period. We can thus use this number to indirectly track the shifting size of the industrial reserve army (e.g. the unemployed population) as our capitalist society grows.

The reader following along in the Desmos app can find $D(t)$ as the first function togglable under the 'other functions relating to the system' grouping (see figure 10).

From this plot, we can see that as long as the warranted paths manage to follow the balanced growth paths, the demand for labor (pictured in green) remains steady and constant at around 0.156, i.e. the work

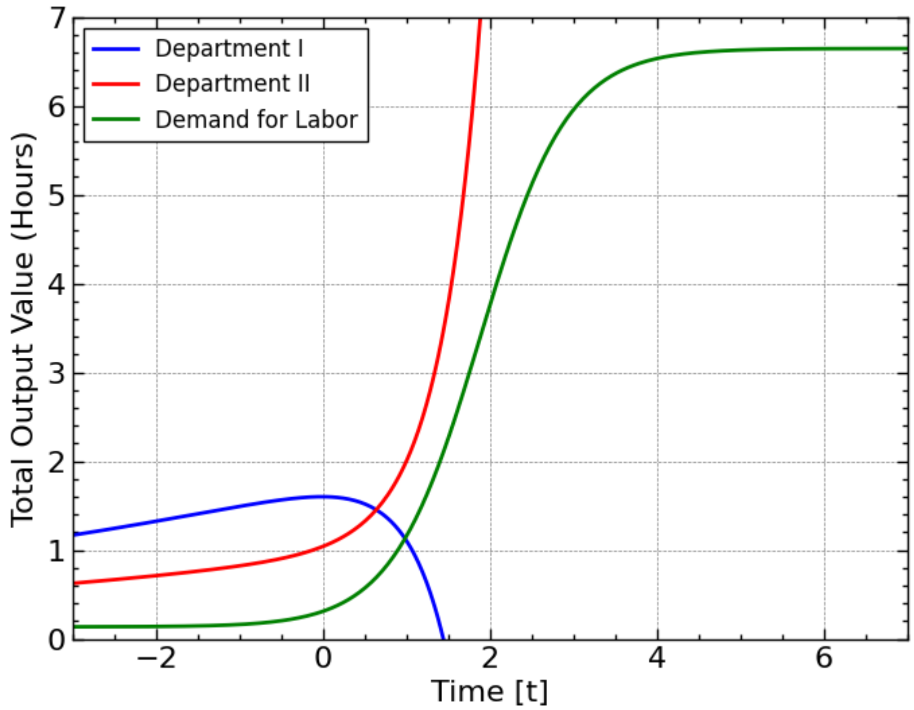

Figure 11: The system if all parameters are held the same as in 10, but k_1 and k_2 are exchanged for one another. Here we see the crisis coinciding with a never-ending hiring spree.

force is steadily increasing at around 15.6% per period. However, as the disruptions begin, we begin to see wild swings in the demand for labor as well. First, we see a slight bump in demand at period 1 (up to 0.2). This is reflective of the de-emphasizing of the more capital intensive wage goods sectors at the cost of the capital goods sectors seen at time $t = 2$.[17] After this, we see demand fall into the negatives between period 1 and 2. By period 2, demand has reached -0.145, meaning that 14.5% of the work force has been laid off, *and did not find work again between periods* 1 *and* 2. These workers were forced to go hungry. Of course, those who survive will find re-employment next period, but this will be as society collapses into it's disproportionality crisis.

In these oscillatory conditions where $k_1 < k_2$, workers who are laid off and forced to go without wages for a period will eventually find work again on the next period, when the favor of capital shines once more on the more labor intensive. We already noted how this is an empty promise due to the fact that they are entering work in the midst of a crisis, but it is also noteworthy that this guarantee evaporates when we allow for technological progress. We will discuss this shortly.

Finally, it should be noted that these crises of disproportionality could coincide with mass layoffs, but they could just as easily coincide with mass hirings. This is especially relevant in the case where $k_1 > k_2$. Figure 11 shows an example. In this system, all parameters are kept the same as in figure 10, except that k_1 and k_2 have been exchanged for one another. Here we see the crisis coinciding with an endless hiring spree. One can also see it coinciding with an endless firing spree if the initial values for y_1 and y_2 are reversed.

6.6 The Overall Composition of Capital

The observations we've thus far made, combined with Marx's other arguments in volumes I and III regarding technological changes, begin to form a 'full story' of how capitalism can be projected to fully immiserate the

[17] This is a bit counter-intuitive at first, but what needs to be understood is that changes in labor employed must *precede* changes in the value outputs. The value outputs, after all, represent the value of finished goods produced in the *previous period*. Thus it would be more appropriate to see fluctuations in the demand for labor as being the cause of the value output fluctuations, not the other way around.

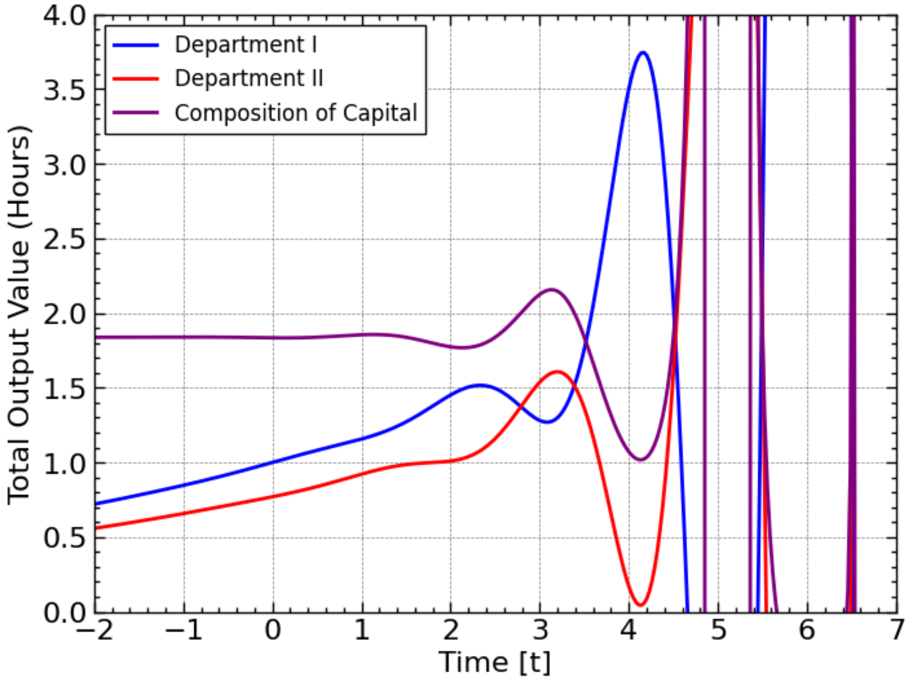

Figure 12: Depiction of the composition of capital.

working class, eventually requiring the expropriation of the expropriators by the working class, and enabled by the independent process of capital centralization. Morishima offers his own accounting of this process. We will look at this first, before I offer my own critiques and minor corrections.

The driving force of Marx's arguments in volumes I and III is Marx's assertion of a rising and in fact acceleration of the overall societal composition of capital over time.

What Marx would call the *value composition of capital* can be easily derived in our model by taking the total value output from both departments which is attributable to constant capital and dividing it by the total value output from both departments which is attributable to variable capital:

$$K(t) = \frac{c_1 y_1(t) + c_2 y_2(t)}{v_1 y_1(t) + v_2 y_2(t)} \tag{45}$$

This function is defined in the Desmos model and can be found and toggled in the 'Other functions related to the system' tab, and is pictured in figure 12.

This is to be contrasted with what Marx would call the *technical composition of capital*, which would be more akin to the k_1 and k_2 values which parametrize our system. These k_i numbers are purely reflective of the methods of production and the current state of labor productivity, and thus the holding fixed of these numbers corresponds to our assumption that these methods are fixed and unchanging.

In addition to the technical and value compositions of capital, Marx also defines the controversial *organic composition of capital* as "the value composition of capital, in so far as it is determined by its technical composition and mirrors the changes of the latter."([7],p.???) Marx's reasoning for defining this third notion is correct: the value composition can change for reasons beyond simply changes in production techniques. We can see how this happens explicitly in our model. In the case of balanced growth, the departmental outputs remain steadily related to each other by a fixed proportion, hence the name. Consequently, as long as warrented growth paths approximate balanced growth, the overall value composition remains fixed as simply a weighted average of the two parameters k_i, weighted according to the unchanging proportional share of total value between the departments (see figure 13). As we leave balanced growth, we can see changes in the value composition which are independent of the technical composition. These disruptions are caused by the favoring of one department over another: if society shifts to favoring the more labor intensive

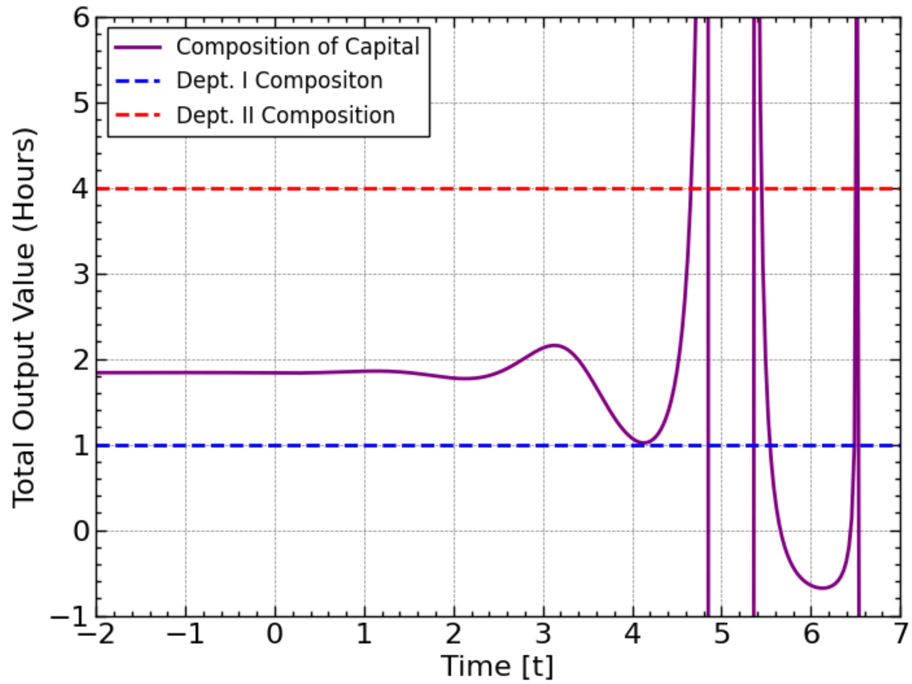

Figure 13: As long as growth remains balanced, the value composition of capital remains a weighted average, bounded between the two departmental composition of capital.

industries, the total value composition of capital will fall, and if it shifts to favoring the more capital intensive industries, then the total value composition will rise.

In defining the organic composition of capital, Marx is attempting to acknowledge the changes we are observing to the value composition which arise out of shifts in the favoring of one industry over another which have differing compositions of capital, and to signal his intent to focus specifically on changes which arise from changes in the methods of production. Marx likely did this because he didn't know what assumptions he needed to make in order to actually ensure that these changes were the only way for the value composition to change. (Ironically, the assumptions he had already made at the end of volume I served to mostly ensure this already!)

In volume I, Marx asserts that increases in the technical composition of capital as a result of labor saving innovations in production due to competition between capitalists would lead to an increase and in fact an acceleration of the organic composition of capital over time. For us, this amounts to the claim that as long as we stick to balanced growth paths, an increase to either of the k_i parameters must correspond to increases in $K(t)$ for all times t. This can be easily witnessed in the Desmos app by setting one of the initial y_{ij} sliders equal to b_{gi} and then adjusting either of the k_i sliders. As we increase either number, the line moves up (see figure 14).

Where this matters to our discussion is in how this overall trend effects the demand for labor, $D(t)$. Predictably, as the technical composition of capital increases, the overall demand for labor along the balanced growth path *decreases*. What we are interested in is not when the $D(t)$ decreases, however. We are also interested in when this value turns negative. Only then do we see an inflationary effect of the reserve army.

In appendix A, I present a mathematical argument to show that the demand for labor $D(t)$ goes negative in response to the overall value composition $K(t)$ undergoes a sudden enough change, with the magnitude of the required sudden change necessary given by the product of the rate of reinvestment with the rate of exploitation ae. Note this is *not* the same as saying that $K(t)$ needs to become sufficiently large or small. The magnitude we need is a magnitude of the change, *not* the magnitude of the value itself. Thus Marx was wise to assert the necessity that the organic composition of capital is not merely increasing but accelerating.

Note the threshold of ae. This can be made sense of by considering what kinds of factors would counteract a tendency for workers to be laid off. A higher rate of reinvestment would (i.e. ae) accomplish this: capitalists

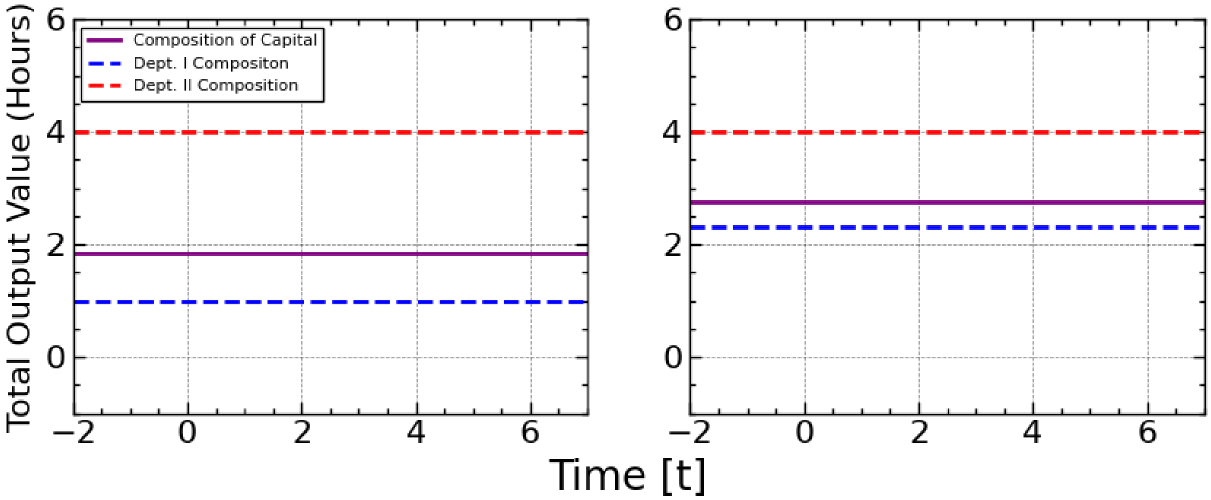

Figure 14: The overall value composition of capital remains bounded between the two departmental compositions (the dotted lines). $K(t)$ remains bounded inside of the rectangle created by the two departmental (technical) compositions. Thus raising the lower k_i coefficient lifts up the composition. Furthermore, raising the upper k_i pulls the value composition up in turn.

investing more means capitalists are hiring more workers. A higher rate of exploitation would also would accomplish this: a more exploited population is more enticing to hire. Morishima notes, as does Marx, that a larger unemployed population created by layoffs brings down the bargaining power of workers and in turn lowers their real wages, raising the value of e. This higher value of e coaxes capitalists into hiring more workers, but further technological changes or layoffs due to disproportionalities between departments would renew the cycle anew, leading to a spiring of exploitation and immisseration.

Morishima writes of this: "As long as k_1 and k_2 have an inherent tendency to increase, or as long as cyclic fluctuations of outputs are unavoidable and explosive, a large reserve army will sooner or later be formed and wages will tend to diminish again. The real wage rate will at last reach its minimal level, below which it is impossible to maintain capitalist production because there will be a rebellion of organized workers. There is thus an upper bound to the rate of exploitation. Similarly an increase in the rate of accumulation, a, will encourage the rate of growth of capital and hence the rate of growth of the demand for labor; but this too is no more than a temporizing policy, because a is bounded by one. The means adopted to overcome crises diminish the means whereby crises can be prevented. At last, capitalists will be at the end of their tether."([7] p.140)

Morishima is saying this in the context of a thorough inspection of Marx's claims about the reserve army in volume I. We can see a clear temptation here from him to bring his own model of the reproduction schema. However, he can't fully do this, as the 'unavoidable and explosive' warranted growth paths only occur when $k_1 < k_2$. Moreover this relationship between k_1 and k_2 is the opposite of what Marx assumed would be the case in a developing capitalist society. Indeed, when $k_1 > k_2$, workers only find themselves experiencing layoffs half of the time in the case that $k_1 > k_2$ (the half in which the department which gradually achieves total supremacy is the more labor intensive one). With all of those disclaimers, it is quite gratifying and instructive to put a bow on our analysis by outlining a crisis cycle that emerges from the special cases where warranted growth paths consistently lead to inflations of the reserve army.

6.7 The Crisis Cycle

It goes as follows. Initially, capitalist society follows warranted growth paths which closely approximate balanced growth. Capitalist society appears prosperous for several cycles, until suddenly, the crisis occurs. The crisis involves a swelling of the reserve army, a corresponding increase in the rate of exploitation e, and the destruction of a large portion of capital. Society finds itself back on what looks like balanced growth paths for a period of time. Workers find themselves being hired again as capital accumulates and is reinvested.

The question is this: will all of the workers who were let go at the start of the previous crisis be rehired before the start of the next crisis? Of course, the answer depends on a variety of factors, most notably the number of cycles which the society is able to continue before another crisis. However, there is a noteworthy evolving property of society which is certainly not operating in the workers' favor here: the increasing composition of capital.

With the assumption of an increasing organic composition of capital (that is to say, assuming changes to the total value composition brought about by changes to production techniques), the new balanced growth path which society finds itself approximating very likely corresponds to a *lower* demand for labor than what was held steady before the crisis hit. Of course, the increased value of e as a result of the inflated reserve army and the increased a as a result of excitement for a new 'boom' cycle act as counterbalances to this, but as Morishima himself notes, there are hard limits to both of these.

Furthermore, if the organic composition of capital is accelerating as Marx claims, then assuming that each crisis resets society to initial values which serve to approximate balanced growth with roughly the same accuracy as before, *it is only a matter of time before the decreases in the demand for labor are consistently preventing the laid off workers from finding new jobs during the next period of the crisis cycle.* The result then, is a reserve army which, from crisis to crisis, grows without bound, leading to an ever-rising rate of exploitation.

We can thus see in our models (with a few extra assumptions) taken along with Marx's arguments from volume I, a vision of the workers increasing immisseration to the point where they will eventually need to overthrow the system, *all within the ideal equilibrium society of the utopian political economists.* At no point are supply and demand ever out of equilibrium. At no point is capitalist society not growing!

To conclude this part of the discussion focused on workers, we pick up where we left off in section 6.2. As subjects of capitalism, we are told that this contradiction between use value and exchange value does not matter. We are told that, while it is true that capitalists are indifferent to the use-values they're producing, the magic of supply and demand will make it so that, approximately, the capitalists will always be producing 'the right' use-values - e.g. the commodities which are actually desired needed. After all, we're told, if capitalists weren't producing use-values which were needed or desired, nobody would buy them, the price would drop, and capitalists would shift to producing something else that actually was wanted. Yet here, we see through a behavior modifying constraint system which exists *independently of supply and demand* that the capitalist class can inadvertantly

The problem with this fairy tale of course is the fact that the only demand that capitalists know how to respond to is *real demand*: demand with the *cash* to back it up. In other words, equilibrium between supply and demand is only maintained because the workers who are laid off - who no doubt have quite a few things to demand for themselves - *are not counted as demand.* The lie of markets meeting demand is the implicit conflation of the *general* demand: the demand associated with the wants, needs and desires of the general population, and demand associated with those who have the means to pay for those wants, needs, and desired. Our society keeps supply and demand in equilibrium, but it does so by casting out a greater and greater section of that population, leaving them to starve.

6.8 A Noteworthy Invariant

One last thing which I find particularly interesting about these paths is how perfectly they emulate balanced growth up until the point where they abruptly they explode off of it. Moreover, the process of *leaving balanced growth*, whether that begins to happen on period $t = 2$ or $t = 20$, always involves the same number of contractions or expansions away from those paths. No matter how you mess around with our system parameters, you will always see one very slight disruption, and then a more pronounced earthquake. After that is an endless sequence of oscillations which are nothing short completely apocalyptic. I believe that this has ideological consequences which are worth considering.

For example, returning to our story, supposing we have reached the end of our crisis with capitalists reasserting power and with new initial value outputs which do good job of approximating conditions for balanced growth, they will have netted themselves a few cycles of what appears to be perfectly balanced growth. This seems to me to be the perfect amount of time for them to reconvince themselves of the viability of market forces. Moreover the abrupt nature of how these paths diverge from balanced growth makes it much easier to blame the disruption on something *other* than market forces. Thus not only is there a crisis cycle which one can see from these results, there is also an ideological mechanism of ensnarement, which after each crisis never allows the capitalists to realize that the system itself is the root causing those crises!

6.9 Marx's Work in More Detail

We are now in a much better position to properly assess the issues with Marx's own reproduction schema. Without loss of generality we will focus on Marx's 'initial schema for reproduction on an expanded scale' (B) found on page 586 of the Penguin edition:

$$\begin{aligned} I \qquad & 4000c + 1000v + 1000s = 6000 && (46)\\ II \qquad & 1500c + 750v + 750s = 3000 && (47)\\ & && (48)\end{aligned}$$

From these numbers we can discern that $k_1 = 4, k_2 = 2$, and $e = 1$, where $y_{1i} = y_1(0) = 6000$ and $y_{2i} = y_2(0) = 3000$, and then from these we can obtain our familiar constants $c_1 = \frac{2}{3}, v_2 = \frac{1}{6}, s_1 = \frac{1}{6}, c_2 = \frac{1}{2}, v_2 = \frac{1}{4}$ and $s_2 = \frac{1}{4}$. As we mentioned, Marx assumes first that capitalists reinvest a fixed proportion $a_1 = \frac{1}{2}$ of their surplus value back into dept. I, and then capitalists from dept. II reinvest whatever proportion $a_2(t)$ of their surplus is necessary for equalizing the demand for capital goods. Equalizing the demand for consumer goods is not as important to Marx because he assumes that capitalists can simply consume the remainder.

With dept. I reinvesting half of it's surplus value, a total of 500 hours, this amount must be split between constant and variable capital in amounts dictated by the composition of capital k_1. With $k_1 = 4$, capitalists in dept. I will be spending four times as much on capital goods as on wage goods. In general, owing back to equation 21 of section 5.2, if the amount X is being reinvested, then the constant capital portion of this investment will be

$$\frac{c_1}{c_1 + v_1} X = \left(\frac{\frac{2}{3}}{\frac{2}{3} + \frac{1}{6}} \right) X = \frac{4}{5} X \tag{49}$$

Thus 400 hours of this 500 hour investment will be spent on constant capital, with the remaining 100 on variable: $500s \rightarrow 400c + 100v$. If the consumer goods dept. were not planning on reinvesting any of it's surplus, then the demand for capital goods would be

$$4000 + 400 + 1500 = 5900 \tag{50}$$

However, the actual output of dept. I is 6000 hours, meaning that dept. II needs to reinvest exactly as much of it's surplus as is necessary to cover this. Whatever the reinvested amount is, it must be that the constant capital component of this equals exactly $100c$. Just as before, with $c_2 = \frac{1}{2}$ and $v_2 = \frac{1}{4}$, we know that the constant capital portion of any investment X is $\frac{c_2}{c_2+v_2} = \frac{2}{3}X$

Thus to find the required investment we simply need to solve

$$\frac{2}{3} X = 100 \Rightarrow X = 150 \tag{51}$$

This investment of 150 hours splits then into $100c$ additional hours of constant and $50v$ hours of variable capital. Thus the capital investments are increased as follows:

$$\begin{aligned} I \qquad & 4000c \rightarrow 4400c, \qquad 1000v \rightarrow 1100v && (52)\\ II \qquad & 1500c \rightarrow 1600c, \qquad 750v \rightarrow 800v && (53)\end{aligned}$$

$$\begin{aligned} I \qquad & 4400c + 1100v + 1100s = 6600 && (54)\\ II \qquad & 1600c + 800v + 800s = 3200 && (55)\\ & && (56)\end{aligned}$$

Accumulation continues the same way in the second year. Of the $1100s$, $550s$ is reinvested, four fifths of which, or $440c$ will be constant capital, and $110v$ will be variable. This leaves $6600-(4400+440+1600) = 160$ hours of constant capital which dept. II must use, amounting to a surplus investment of 240 hours. Thus we have the change in capital investments:

$$I \qquad 4400c \rightarrow 4840c \qquad 1100v \rightarrow 1210v \tag{57}$$

$$II \qquad 1600c \rightarrow 1760 \qquad 800v \rightarrow 880v \tag{58}$$

At the end of the year then, value outputs will be

$$I \qquad 4840c + 1210v + 1210s = 7260 \tag{59}$$

$$II \qquad 1760c + 880v + 880s = 3520 \tag{60}$$

In order to make sense of this, we should think about this growth in terms of equations. Back in section 5.2, we saw that in a situation where capitalists were reinvesting the proportion a of their surplus value, *and* were committed to reinvesting strictly in their own departments, then equation for $y_i(t+1)$ could always be written

$$y_i(t+1) = (1 + a\pi_1)y_i(t) \tag{61}$$

where $\pi_i = \frac{s_i}{c_i+v_i}$ is the departmental value profit rate. We have both of those situations here, and so these equations will suit us well. For department I, we have

$$a\pi_1 = \frac{1}{2}\frac{\frac{1}{6}}{\frac{2}{3}+\frac{1}{6}} = \frac{1}{10} = 0.1 \tag{62}$$

$$\Rightarrow y_1(t) = (1.1)^t y_1(0) \tag{63}$$

Plugging in $t = 1$ gives us that $y_1(t) = 6600$, exactly as we calculated. As we can see, dept. I's growth is completely independent of dept. II, and this is because dept. II is disallowed from investing it's own surplus into dept. I. However, dept. I's growth is not independent of dept. II, since it's surplus reinvestment rate a_2 depends on dept. I. We need to calculate this number, and then find a formula for it in order to identify the unintended side effect that it had for Marx. Additionally, we should expect it to vary with time, so that our $y_2(t)$ equation looks like

$$y_2(t) = (1 + a_2(t)\pi_2)^t y_2(0) \tag{64}$$

After calculating $\pi_2 = \frac{1}{3}$, we can easily solve for $a_2(0)$ since we already know $y_1(0)$ and $y_2(0)$:

$$y_2(1) = \left(1 + \frac{1}{3}a_2(0)\right) y_2(0) \tag{65}$$

$$\Rightarrow 1 + \frac{1}{3}a_2(0) = \frac{y_2(1)}{y_2(0)} \tag{66}$$

$$\implies a_2(0) = 3\left(\frac{y_2(1)}{y_2(0)} - 1\right) = 3\left(\frac{3200}{3000} - 1\right) = 0.2 \tag{67}$$

Thus, comparing the growth rates of the two departments, dept. I reinvested 50% of it's surplus in order to grow from 6000 to 6600 amounting to a growth rate of $a_1\pi_1 = (0.5)(0.2) = 10\%$, and dept. II reinvested 20% of it's surplus to grow from 3000 to 3200, amounting to a growth rate of $a_2(0)\pi_2 = (0.2)\frac{1}{3} \approx 6.67\%$. If we were to perform the same calculations going from year 1 to year 2, we would find that during this period, dept. II is reinvesting

$$a_2(1) = 3\left(\frac{3520}{3200} - 1\right) = 0.3 \tag{68}$$

Meaning that they are reinvesting 30% of their surplus and growing at a rate of $(0.3)\frac{1}{3} = 10\%$ - the same as dept. I. If we look at Marx's results for the following year:

$$I \qquad 5324c + 1331v + 1331s = 7986 \tag{69}$$

$$II \qquad 1936c + 968v + 968s = 3872 \tag{70}$$

we would find

$$a_2(1) = 3\left(\frac{3872}{3520} - 1\right) = 0.3 \tag{71}$$

so that once again, both departments are growing at a rate of 10%. We would find the same results if we looked at years 3 or 4 (which is where Marx stopped). After the first year, we have the case where depts. I and II are reinvesting their surplus values at different rates (50% in the case of I, 30% in the case of II), *but are nonetheless growing at the same rate.* This is balanced growth!

We can see now more clearly the hidden side effect which Marx's investment scheme had. By forcing capitalists in dept. II to reinvest exactly as much as was necessary to facilitate dept. I's desired growth, this had the unexpected effect of snapping dept. II's output to one which allowed for balanced growth between the departments.[18] After this point, the system stabilized, and Marx was unable to witness the results we were able to.

Marx had no reason to assume that his investment scheme would preclude him from witnessing unbalanced growth. Indeed, the growth he witnessed initially was *not* balanced. However, as we've seen already in our own model, balanced growth is often a function of the initial conditions. It turns out that the output one gets from the unbalanced growth perfectly positions the both departments to grow at a steady rate after this point.

6.10 Conclusions

Let us now summarize the conclusions which we are proposing Marx would have come had he been able to properly carry out his reproduction schema in the way we are claiming he actually meant to. In line with Marx's dual intensions, we must think about these conclusions two dimensionally. We must consider what conclusions are worth drawing from the standpoint of a critique of political economy, and from the standpoint of laws of motion of capitalism.

Starting with the former category, the political economists of Marx's time believed that when capitalism was allowed to settle into a 'steady state' wherein market relations drove production in their entirety and the rate of profit had been fully equalized across all industries, investment decisions would become a pure function of supply and demand. This emergent correspondence would, according to them, resolve the contradictions between use-value and exchange-value, synchronizing them so that all choices made by capitalists in the sphere of production, while being made on the basis of maximizing exchange-value revenue, would nonetheless always be reflective of the shifting use-value desires and needs of the overall population. At this point, the profit seeking commodity production system of capitalism would become an effective, albeit imperfect, regulator of production and distribution. Nobody would get everything they wanted. Indeedn, some might get more than they need, and others may even fall through the cracks. On the whole, however, market mechanisms would drive societal growth while maximizing prosperity, rationally coordinating production according to people's ever shifting wants, needs an desires, and rationing that output in a fair way via changes in prices.

Beliefs such as these plague our discourse to this day, as they are the essential foundation to all libertarian belief systems. Anyone who argues that the problems of society are due to a lack of sufficiently 'free markets',

[18] See *Marx's Economics* page 120 for a general proof that this is specifically due to his investment scheme.

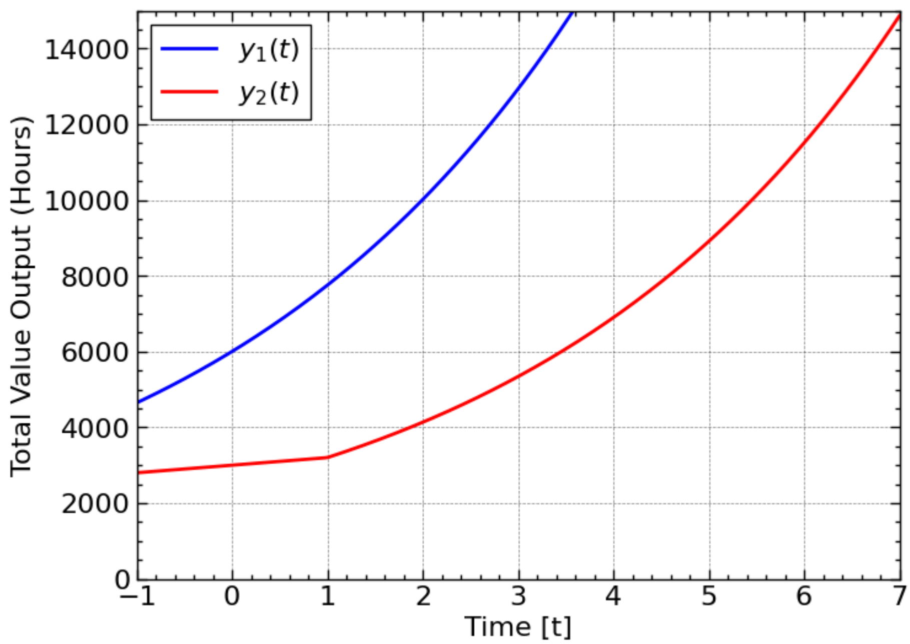

Figure 15: Plotting Marx's actual growth curves from Volume II. Note the jagged edge: Marx's investment scheme effectively forces dept. II's output to snap to a number which 'plays nicely' with dept. II. After this initially unbalanced action, steady balanced growth is assured.

blaming the problems of our society on over-regulation, implicitly *must* believe that market driven capitalist production works in this way.

Marx questioned this belief. In Volume II he asks the question: what does such a pure and perfectly functioning capitalist society in perpetual equilibrium actually look like? What comes out of such a society in terms of actual use-values? What does such a growth trajectory actually look like?

Had Marx been able to carry out the mathematical methods necessary for evaluating his own system in full, the answer which he would have found is simply this: chaos. He would have shown that even if nature and labor are perfectly obedient and always available for purchase, and even in the presence of perfect competition, perfect information, perfectly efficient markets, and perfect capital mobility, there would still nonetheless be periodic crises of disproportionality. These perfections do nothing to address the fundamental problem of growth bounded by it's own output of finished goods. Such limitations, we have seen, create the perverse incentive for *divestment* - perverse because they are forces independent of any shifting of public demand for use-values. Such uncompromising lust for economic growth in the face of resource constraints and differences in compositions of capital between industries drives even an ideal society such as that of our model into a wildly disruptive feedback loop, in which workers are routinely laid off in ever-increasing numbers. These layoffs, in some conditions dictated by the composition of capital, just as routinely leave workers unemployed for entire production periods. Indeed, the reserve army can find itself inflating to a point of mass immisseration, or deflating to the point of a labor shortage crisis.

With each successive crisis of disproportionality taking place in this conjectured capitalist paradise, the capitalist class will attempt, through a mass act of creative destruction of existing capital, to recalibrate themselves to a balanced growth path. However, our completed model exposes that this task is futile. The balanced growth path is fundamentally unstable, and statistically impossible to fully achieve. Without impossibly perfect accuracy, the crisis *will* eventually arrive, abruptly and with little warning. An abrupt end implies a peaceful beginning, and indeed the periods preceding our crises will seem relatively prosperous, as the growth appears steady balanced. These periods of prosperous growth are just long enough for the capitalist class to reconvince themselves of the soundness of market mechanisms, and shift the blame somewhere else when the disruptions finally arrive. Locked into such growth paths, the capitalists will thus *likely never learn their lesson.*

Before continuing to discuss conclusions for actual existing capitalist systems, it should be emphasized that caution is needed, as these crises of disproportionality we have been discussing are *entirely theoretical*, occuring within a very unrealistic 'toy model'. A great deal of justified criticism has been levelled at the so-called neo-Ricardian equilibrium modelling tradition in recent years. The unrealistic premise makes it difficult and dangerous to interpret the dynamics obtained within such models directly onto the real world. Nonetheless, if we are careful, there are real conclusions which can be drawn from it. However, these real conclusions can only follow from the most realistic version of the model. Thus we suspend our analysis here until the next section.

7 The Dazzling Money Form

Smith, Ricardo, and Marx all understood that in a developed capitalist system, prices and labor values could not be directly proportional to one another. Smith was the first to make this observation, noting that while prices and labor values must necessarily coincide in a so-called 'early and rude state of society', they must necessarily diverge as capitalism developed. This was informed primarily by the belief in a gradual approach towards equilibrium profit rates. We already noted (where?) that a direct proportionality labor theory of value is incompatible with the notion of an equilibrium profit rate, since if prices and labor values coincide, more labor intensive industries will be more profitable than less labor intensive industries.

There is a tension between the belief, on the one hand, that natural prices serve to quantify the difficulty of production, and on the other hand the belief in capitalist production approaching equalized profit rates. If one wants to argue that the former exists in the presence of the latter, then it falls on them to explain precisely how prices can be derived from these costs, or conversely how these costs (if properly defined) can be *transformed* into natural prices. Within the history of political economiy, this emergent dialectical contradiction became known as the transformation problem.

Ricardo struggled a great deal with this problem. He had no special affinity towards labor, merely seeing it as a good choice of *numeraire* for measuring resource costs as a scalar number. Ricardo's merely sought to clarify the particular manner in which prices reflected difficulty of production in the presence of uniform profit rates. Marx inherited this unresolved problem from Ricardo, but his intensions were very different. While Marx *did* attempt to describe a process of transforming values into equilibrium prices of production, this was only the means to an end for him. Marx's real aim in approaching the transformation problem was twofold. On the one hand, he sought to demonstrate how surplus value still remained the sole source of profit during and after such a process. On the other hand, he sought to demonstrate how this very divergence of prices from values serves an essential purpose to the stability of the capitalist system.

Unlike other class societies, capitalism takes on an objective character to its subjects. Buyers and sellers meet freely in the market and exchange goods and services at prices agreed upon by both parties. No value is extracted through violent coercion. Instead, value is extracted through the disparity between the price paid for a day's labor and the prices paid for the goods produced. Marx understood that if this disparity were too clearly visible to the subjects of capitalism, both worker and capitalist alike, then the system would not be viable. These disparities must be hidden behind a veil of prices.

More broadly, the idea that material conditions must somehow feed back into the ideology of it's subjects in a way which obscures the exploitative character of the system and subsequently stabilize it is found all over the three volumes of capital, but is most prominent in volume III. For example, of the rate of exploitation and it's relation to the rate of profit, Marx writes: "the rate of surplus profit is from the very outset distinct from the rate of surplus value... But... this serves, also from the outset, to obscure and mystify the actual origin of surplus-value, since the rate of profit can rise or fall while the rate of surplus value remains the same, and vice versa, and since the capitalist is in practice solely interested in the rate of profit."

To Marx, the transformation process, in scrambling prices and distancing them from their values, is the key factor which imbues money with this 'dazzling', or 'obfuscating' character. Marx writes: "The transformation of values into [different] prices of production serves to obscure the basis for determining value itself." Additionally, it serves to de-individualize the class relations, transforming exploitation from an individual relation between employer and employed into a social relation between the working class as a whole and the capitalist class as a whole. Of this he writes: "The individual capitalist (or all the capitalists in each individual sphere of production), whose outlook is limited, rightly believes that his profit is not derived solely from the labor employed by him, or in his line of production. This is quite true, as far as his average profit is concerned. To what extent this profit is due to the aggregate exploitation of labour on the part of total social capital, i.e. by all his capitalist colleagues - this interrelation is a complete mystery to the individual capitalist; all the more so, since no bourgeois theorists, the political economists, have so far revealed it." From this textual evidence, we can clearly see two things clearly. First, that Marx views money, or more generally the value-form, as possessing a 'dazzling', or obfuscating character which has far-reaching ideological effects on it's subjects and which serve to stabilize social relations. Second, that the transformation process is a key factor in where money obtains this character.

In this section, we will be able to observe very clearly using our model the issues with Marx's proposed solution to the transformation problem, which have led to such great controversies over the course of the 20th century and into the present day. We will also be able to witness a silver lining, the so-called Fundamental Marxian Theorem - that that despite these flaws, surplus value nonetheless remains a necessary and sufficient condition for profit. Most interestingly of all, we will be able to witness something new - and through it verify that Marx's claim about the secret role of money - as agent of obfuscation - is far more correct than he could likely anticipated.

7.0.1 Witnessing the Transformation Problem

Marx's conception of the transformation process, e.g. the process of profit rates equalizing across all capitalist firms, takes the implicit form of a intra-class struggle among the capitalist class, with consumers caught in the crossfire. Capitalists in the less profitable industries either move into more profitable sectors and create supply gluts, bringing down the prices of more profitable goods below their natural levels, or simply jack up the prices of their less profitable goods to a level they see as fair. As supply and demand is disrupted in this process, consumers react by adjusting their consumption habits to conform with the shifting price realities created through the struggle. This active redefining of social demand synthesizes with the changing supply, as prices continue to diverge from their original values. Such a process can only stabilize with a resolution to this intra-class struggle, and this requires a uniform profit rate to prevail, so that all capitalists enjoy their 'fair share' of profit per dollar of investment. Marx assumes that this uniform profit rate is in fact the *average* rate of profit, and moreover that this number is an invariant of the process. Thus the average rage of profit, determined initially in terms of technology and exploitation,

$$\pi = \frac{S}{C+V} \tag{72}$$

becomes the uniform rate of profit at the end of it, even in the absence of any remaining correspondence between prices and values. (Here S, C and V are the total surplus value, constant, and variable capital associated with the whole of society over the course of some fixed time period.)

At the end of the process, with a uniform rate of profit established, all capitalists find themselves earning equal returns on their investments. Using this, Marx gives an example of calculating the newly emergent prices of production. He estimates the *cost-price* of a commodity directly their values $c_i + v_i$, where c_i and v_i are the values of the means of production and of labor power respectively which are necessary for producing a unit of commodity i, and simply scales this up according to the rate of profit. In other words, prices of production are defined

$$p_i = (1+\pi)(c_i + v_i)$$

As Marx remarks, some of these prices will be higher than their values, while others will be lower. Nonetheless, owing to π being the average rate of profit across all sectors, all deviations in prices from their values add up to 0. Subsequently, Marx finds that at the end of the production process, the following three 'conservation laws' remain in effect.

At the end of the process, Marx asserts that despite the individual prices of commodities diverging from their labor values, it is nonetheless still the case that

(1) The uniform rate of profit which prevails across all sectors is precisely the overall value rate of profit.

(2) The total price of all commodities equals the total labor value of all commodities. [19]

(3) The total surplus value produced by society must still be directly proportional to the total profit earned by capitalists, with the same proportionality constant as item 1. [20]

Taken together, these 'conservation laws' together assert that surplus vaue remain the sole source of all profit, even after the equalization of profit rates. Consequently, the scrambling of prices from labor values merely reflects a *redistribution* of that surplus value among the capitalist class, such that each capitalist, regardless of industry, obtains their *fair share* of profit, directly proportional of the size of their capital investment.

Marx himself is quick to note that his own formula for calculating production processes is only a first approximation, since the cost price used $c_i + v_i$ are values and not prices of production themselves. Marx implies, however, that upon finding all of the p_i values, one would have found, among other things, a better estimate of the cost price $c_i' + v_i'$. These could be used in a repetition of the same calculation: $p_i' = (1+\pi)(c_i' + v_i')$. Marx is therefore merely describing the first step of an iterative procedure which he conjectures will converge to the actual cost prices.

Before proceeding we should address is the issue of units. Prices are measured in dollars, while values are measured in units of time - typically hours. This entire iterative procedure, and claims such as conservation laws 2 and 3 above, only make sense if we are assuming some kind of unit conversion has already taken place which makes prices and values comparable. There are two workarounds here, which are often confused with one another. The fisrt is to express dollar values not in units of dollars, but rather in terms of the number of hours of labor which could be *purchased* (or *commanded*) using those dollars. The second is to express dollar values in terms of the number of hours of value which could be *produced* using those dollars.

These two ideas are not the same, and the reason is exploitation. Say for the sake of example that the rate of exploitation is 1. Then for every hour worked on average, thirty minutes of value will be returned to the worker in real wages, while thirty minutes of surplus living labor will be provided to the capitalist. However, the capitalist only pays the wages. Thus the labor commanded by a dollar will always be an inflated metric, since it undervalues labor by a factor represented by the rate of exploitation. This distinction was not well understood until recently with the work of the so-called New Interpretationists such as Duncan Foley, Gérard Duménil, and many others, with their work on the MELT, or monetary expression of labor time[3][11][14].

Morishima himself seems unaware of this distinction, and employs the former conversion method. In particular, he divides dollar amounts by the hourly wage to convert to hours of labor *commanded*. If p is the dollar price of a commodity, and with w the hourly wage in dollars, we will let $p_w = \frac{p}{w}$ denote the number of unit wages purchaseable (or commanded) by p many dollars. The units of p_w are hours, just like values, and thus prices and values have become directly comparable.

On the other hand, New Interpretationists define the MELT, commonly denoted τ, as the ratio of the price of the net product divided by the value of the net product[21] Effectively, this expresses the number of 'dollars' of value produced per hour of labor worked. Note that the units of the MELT are dollars over hours, just like the unit wage w. Thus dividing a dollar value by this number serves to convert dollars to hours in a different way not by measuring the number of hours of wage labor which would be purchaseable with those dollars, but rather by measuring the number of hours of labor worked which would produce goods selling for those dollars (averaged over the whole of the economy).

Controversies abound with respect to this overall presentation. Many will no doubt take issue with various aspects of my brief summary. However, the crucial thing which we must note before all else is that all of Marx's results here follow from the *assumption* that the equilibrium rate of profit experienced by capitalists, which we will denote π_e, equals Marx's overall value rate of profit $\pi = \frac{S}{C+V}$. Indeed, it can be shown (with a few extra assumptions[22] that Marx's proposed method for calculating production prices (interpreted as

[19]"The sum of all prices of production of all commodities produced in society - the totality of all branches of production - is equal to the sum of their values."

[20]"Surplus value and profit are identical from the standpoint of their mass."

[21]By net product we mean the total bundle of finished goods. In our model, the value of this bundle is $x(t) = y_2(t) + (y_1(t) - c_1 y_1(t) - c_2 y_2(t))$. The price of this bundle can be found after solving for prices, which we proceed to do in the next subsection.

[22]namely, it must be the case that the augmented input-output matrix M representing the total bundle of capital and wage goods required for the production of a unit of each commodity is *primitive*. In economic terms, this amounts to the claim that the capital goods industries are tightly dependent on one another, that luxury goods do not exist, and that at least one wage goods industry requires living labor input. See for example Morishima (1978) p. 163[12].

an iterative procedure in the way we've described) is guaranteed to work provided that $\pi_e = \pi$. Since this assumption is also what allows him to conclude 2 and 3, we can see that his entire argument is contingent on this one assumption. We begin here then: is the equilibrium rate of profit truly equal to the overall value rate of profit? Let us investigate this within our model.

To start, we need to consider prices within our model. Let p_1 denote the price of an hour's worth of output from department 1 and p_2 denote the same for department 2. Consider the task of producing an hour's worth of capital goods output. This requires the purchasing of c_1 hours worth of capital goods and v_1 hours worth of wage goods. Thus the necessary costs for the capitalist in producing an hour's worth of capital goods is $c_1p_1 + v_2p_1$. With π being the uniform profit rate, it must follow that the price of the produced output equals this necessary cost plus a proportion of it as profit: $\pi(c_1p_1 + v_2p_1)$. This, along with the same observations made for department 2, give us the system of equations

$$p_1 = (1+\pi)(c_1p_1 + v_1p_2) \tag{73}$$
$$p_2 = (1+\pi)(c_2p_1 + v_2p_2) \tag{74}$$

Note that in this system we have two equations but three unknowns. Thus it would seem like we have an underdetermined system with a very flexible set of solutions. Appearances can be deceiving, however. First, let's acknowledge and discard with the trivial solution: if $p_1 = p_2 = 0$, then both equations are satisfied, for any rate of profit π, but this is clearly not what we are looking for. If we rule this out, and additionally require that both prices be possitive, then it turns out that there is only a single viable rate of profit[23]:

$$\pi_e = \frac{1}{\frac{1}{2}\left(c_1 + v_2 + \sqrt{(c_1 - v_2)^2 + 4c_2v_1}\right)} - 1 \tag{78}$$

The reader is discouraged from dwelling on the actual formula here - they just need to understand that it is fixed and can only equal this one number. Many possible price vectors accompany this. This leaves us with two equation and two unknowns, so that p_1 and p_2 can be solved for. There are an infinite number of possible solutions for p_1 and p_2. One such choice could be

$$p_1 = \frac{1}{1+\pi_g} - v_2 \tag{79}$$
$$p_2 = c_2 \tag{80}$$

Any constant multiple of these two numbers would also solve our system. Seeing the prices as an arrow in $\mathbb{R}^2$ aimed at the point $(\pi_e - v_2, c_2)$, we are saying that this arrow can be arbitrarily bigger or smaller, so long as it stays pointed in the same direction. Putting these prices aside for a moment, let us consider this equation we have derived for the equilibrium rate of profit π_e, and compare it to Marx's value rate of profit within our model. At time t, the total surplus value output is $s_1y_1(t) + s_2y_2(t)$, while the total capital output is $(c_1 + v_1)y_1(t) + (c_2 + v_2)y_2(t)$. Thus Marx's rate of profit is

$$\pi(t) = \frac{s_1y_1(t) + s_2y_2(t)}{(c_1 + v_1)y_1(t) + (c_2 + v_2)y_2(t)} \tag{81}$$

[23] To see this will require a bit of linear algebra. Let $a = \frac{1}{1+\pi}$. Substituting this in for π and distributing and rearranging the terms gives us the system

$$(c_1 - a)p_1 + v_1p_2 = 0 \tag{75}$$
$$c_2p_1 + (v_2 - a)p_2 = 0 \tag{76}$$

In matrix form this is

$$\begin{pmatrix} c_1 - a & c_2 \\ v_1 & v_2 - a \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow (M - aI)\vec{p} = \vec{0} \tag{77}$$

For $\vec{p}$ to have a nontrivial solution, the matrix $M - aI$ must therefore be singular, so that it has a nontrivial nullspace. This is true if and only if the determinant of $M - aI$ is equal to 0, i.e. $|M - aI| = 0$. This equation does not involve p_1 and p_2; we just have a single equation with a single unknown, allowing us to solve for a and in turn π.

From these two equations 78 and 81 we can immediately see an issue. Equation 81 is a function of time, while equation 78 is not. Marx tended to see his value rate of profit as only changing in response to changes in the overall composition of capital or changes in the state of the class struggle. What he may have failed to anticipate - and this is easily overlooked if one is working within in a single department model, as he was frequent to do in volumes I and III - is that his overall value rate of profit can also shift with the shifting proportional emphasis between different sectors and departments. To see this more clearly, consider the departmental value rates of profit:

$$\pi_1(t) = \frac{s_1 \cancel{y_1(t)}}{(c_1+v_1)\cancel{y_1(t)}} = \frac{s_1}{c_1+v_1} \tag{82}$$

$$\pi_2(t) = \frac{s_2 \cancel{y_2(t)}}{(c_2+v_2)\cancel{y_2(t)}} = \frac{s_2}{c_2+v_2} \tag{83}$$

Here, as long as wages and the techniques of production remain constant, we can see that the departmental rates of profit are independent of the mass of their output, and thus are constant with respect to time within our model. If all sectors of the economy have been aggregated into a single department, as Marx tends to assume outside of volume II, then this phenomena would be seen for the overall value rate of profit as well. Turning towards volume II, we have already noted that with his investment scheme, Marx inadvertantly locked himself into only witnessing balanced growth. In a multi-department model, balanced growth is precisely what is needed in order to hold his value rate of profit constant with time. Suppose that $\frac{y_1(t)}{y_2(t)} = \gamma$ for some γ. Then

$$\pi(t) = \frac{s_1 y_1(t) + s_2 y_2(t)}{(c_1+v_1)y_1(t) + (c_2+v_2)y_2(t)} = \frac{s_1 \gamma y_2(t) + y_2(t)}{(c_1+v_1)\gamma y_2(t) + (c_2+v_2)y_2(t)} \tag{84}$$

$$= \frac{(\gamma s_1 + s_2)\cancel{y_2(t)}}{(a(c_1+v_1) + (c_2+v_2))\cancel{y_2(t)}} \tag{85}$$

$$= \frac{\gamma s_1 + s_2}{a(c_1+v_1) + (c_2+v_2)} \tag{86}$$

Thus when growth is balanced, the overall rate of profit is a weighted average of the departmental rates of profit. However, when growth is not balanced, as is typical in our model, the overall composition of capital can vary drastically with time as we saw in section 6.6. All we are noting here is that the overall value rate of profit change in in turn with this. If society shifts away from the more labor intensive department towards the more capital intensive department, then the overall composition of capital will rise, and the overall value rate of profit will fall. Figure 16 shows all four of the value rates of profit we just discussed plotted together. Just as we saw with the compositions of capital, the overall value rate of profit is a weighted average, hovers inside of the rectangle given by the two departmental rates. Moreover, to the extent that the warranted growth paths manage to approximate balanced growth, the overall actual balanced growth path in turn approximates the rate of profit associated with a balanced growth path, but as disruptions and disproportions occur, it flies off of this line.

The idea that the value rate of profit can change with time based on which departments are being emphasized and which are not is clearly incompatible with it's supposed function as a uniform rate of profit experienced in equilibrium; if capitalists are experiencing a constant and unchanging money rate of profit, then there is no way that this could possibly be our $\pi(t)$. However, this does not necessarily damage Marx if we assume he was not talking about the *actual* value rate of profit, but rather about the balanced growth path value rate of profit which the actual growth path is emulating. Thus, while noteworthy, this isn't really the beating heart of the controversy. The real issue comes from observing that *even if* we go with the balanced growth path rate of profit, it still cannot in general equal the equilibrium rate of profit given by equation 78.

At this point, to make things more concrete, let's introduce some example numbers. We will go with the same numbers we used before in section 5: $c_1 = \frac{1}{3}$, $c_2 = \frac{2}{3}$, $v_1 = \frac{1}{3}$, and $v_2 = \frac{1}{6}$. Plugging these numbers into equation 78 gives us $\pi_e \approx 0.372$. If we assume balanced growth, then the applet would show us that this requires $y_2(t) \approx 0.775 y_1(t)$, so that by equation 84 we have $\pi \approx 0.352 \neq \pi_e$. Thus, even under conditions

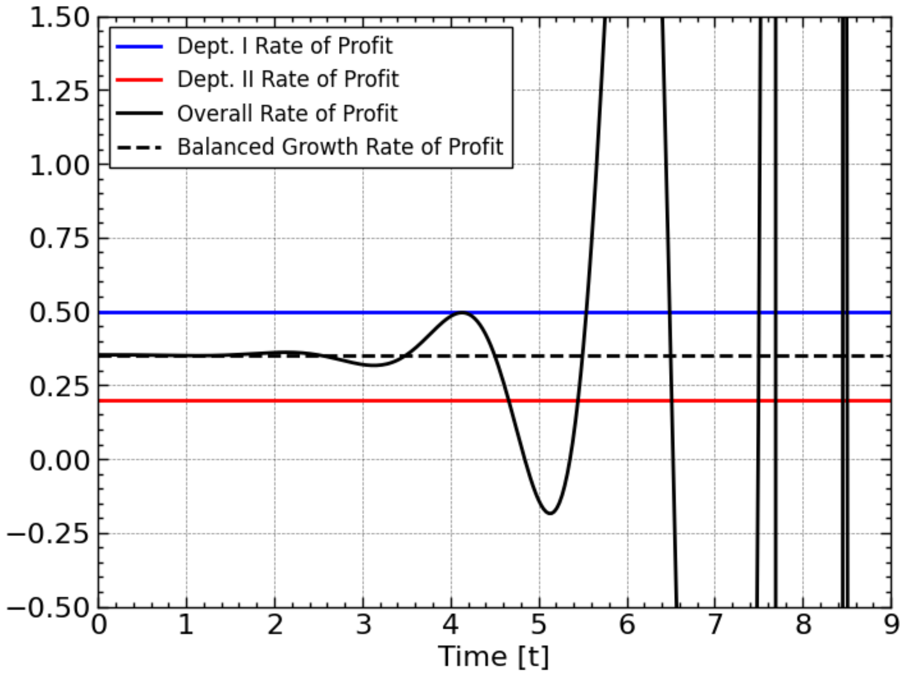

Figure 16: Value rates of profit. Blue and red lines are the rates of profit for departments I and II respectively. Solid black line is the value rate of profit. Dotted black line is the rate of profit on the balanced growth path which the actual warranted path approximates.

of balanced growth, Marx's overall value rate of profit cannot, in general, equal the equilibrium money rate of profit experienced by capitalists. This result is really the core of the transformation problem - where it becomes a problem for Marx.

Why did this happen? Why can't it be the case that the overall value rate of profit equals the actual equilibrium rate of profit? On a basic level, the reason is simply because the *values* of commodities are a purely technical set of numbers - they are fixed by the tools and methods of production which society employs. Marx's extremely valuable insight throughout his analysis of capitalism is that the technology available in a society profoundly constrains that society's social relations.[24] There is a gulf of difference, however, between constraining and outright determining. In particular, it is clear that the *state of the class struggle* also has an influence on both prices and the rate of profit. This can be seen clearly by inspection of the system of equations 79. The presence of v_1 and v_2 here gives away the game here, as v_i is merely the component of an hour of living labor which goes to the worker in the form of wages, rather than the total labor itself. Without any change to technology, if workers secure higher wages, then this changes v_i without changing l_i, the actual living labor required for producing something. Marx understands this limitation of the value theory well - it is a primary motivation for why he takes the time to explain the transformation process in the first place. His mistake was in not recognizing that it is not just prices that diverge from values because of this overdetermination, but the money rate of profit from the value rate of profit as well.

7.0.2 Total Prices vs Total Value, Total Profit vs Total Surplus Value

We have seen that Marx's overall value rate of profit, even on a balanced growth path where it remains constant, cannot in general equal the actual uniform money rate of profit experienced by capitalists in our equilibrium model. It follows that Marx's iterative method for converting values into prices of production will not succeed in general. What of his conservation claims 2 and 3? Picking up where our numerical

[24] Marx himself writes in *The Poverty of Philosophy*: "Social relations are closely bound up with productive forces. In acquiring new productive forces men change their mode of production; and in changing their mode of production, in changing the way of earning their living, they change all their social relations. The hand-mill gives you society with the feudal lord; the steam-mill, society with the industrial capitalist."[8]

example left off and utilizing our price equations 73, we would have that the price of an hour of output from department I $p_1 \approx 0.56$ dollars, and $p_2 \approx 0.67$ dollars. However, we noted before it would be both nonsensical and unfair to Marx to compare prices in dollars directly with labor values. As mentioned previously, the way in which this normalization was done during Morishima's time period was to view an amount of money as an amount of time by measuring how many hours of labor could be purchased with that money (e.g. measure the hours of labor *commanded*). Now, a more modern approach has emerged using the MELT, but to start things off we will go with the more classical approach. The unit prices for an hour of output from each department [25] is expressed as a quantity of labor time commanded by dividing by the hourly unit wage w (which we do not currently know). Let $p_{1w} = \frac{p_1}{w}$ and $p_{2w} = \frac{p_2}{w}$. All three of these numbers can be calculated without too much trouble using the tools that we already have.

The key to solving for the hourly wage in dollars in our model is noting that there is only a single thing which the worker will purchase using their wages - the wage good, which has unit price p_2. Note that $\frac{w}{p_2}$ thus represents the real hourly wage of a worker - it is the number of units of the wage good which the worker will be able to purchase with an hour's wage. However, throughout this paper I have chosen for the sake of simplicity to conflate the real units of an output (e.g. number of tires, bushels of wheat, etc) with time output. The 'unit' of real output from our two departments, throughout this paper, has been and will continue to be one hour of generic value. Thus $\frac{w}{p_2}$ is not only the number of units of the wage good which a worker can purchase with an hour's wage, it is also the *value* of that number of units, since these are one in the same. On the other hand, when the capitalist purchases an hour of labor power, they receive one hour of labor. Thus the surplus labor time which the capitalist receives for their purchase is $1 - \frac{w}{p_2}$. Consequently, the rate of exploitation is the ratio of this surplus time to the value of the means of subsistence:

$$e = \frac{1 - \frac{w}{p_2}}{\frac{w}{p_2}} \tag{87}$$

But $\frac{w}{p_2} = \frac{1}{p_{2w}}$. Thus

$$e = \frac{1 - \frac{1}{p_{2w}}}{\frac{1}{p_{2w}}} = p_{2w} - 1 \tag{88}$$

and so $e = p_{2w} - 1$. Therefore

$$p_{2w} = e + 1 = 2 \tag{89}$$

Note that this does not depend on the actual equilibrium prices at all! This makes sense only once we remember that we've defined the unit of output for commodity 2 to simply be an hour of value. What this equation says in lieu of this is that the number of hours of labor purchaseable by the price of an hour of value in commodity 1 is one hour plus whatever proportion of unpaid labor is dictated by the rate of exploitation. From this, we also can calculate the hourly wage in dollars:

$$p_{2w} = \frac{p_2}{w} = e + 1 \Rightarrow w = \frac{p_2}{1+e} \approx 0.34 \text{ dollars} \tag{90}$$

Finally, using this we can solve for p_{1w}:

$$p_{1w} = \frac{p_1}{w} = \frac{p_1}{\frac{p_2}{1+e}} = (1+e)\frac{p_1}{p_2} \approx 1.68 \text{ hours} \tag{91}$$

We are now in a position to evaluate Marx's claims 2 and 3 within our model. Starting with 2, the total value output at time t is simply $y_1(t) + y_2(t)$. The total price (measured in number of unit wages purchaseable) of this is

[25] Implicitly, we are seeing a unit of output from each department as being essentially 1 hour of value. This abstraction arises from aggregating multiple sectors into a single department. Alternatively, the reader can think of our model as a simply two commodity economy with a single wage good and a single capital good, each of which has a unit value of 1. Both interpretations are equally valid.

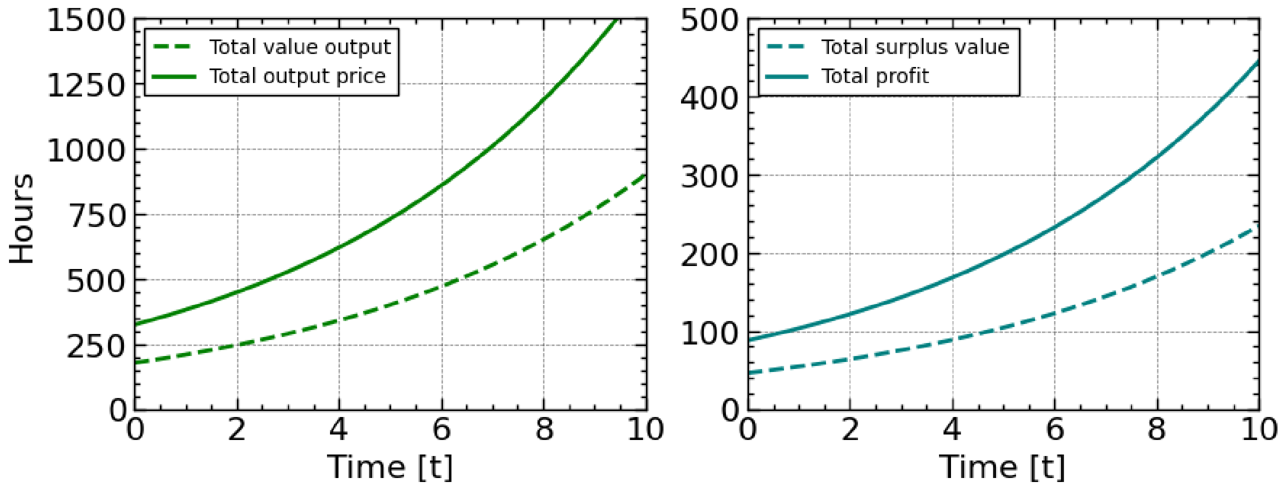

Figure 17: Comparing total prices and profits with total value and surplus value respectively. Here, $y_{1i} = 100$, y_{2i} is set to ensure balanced growth, $k_1 = 1, k_2 = 4$, and $e = 1$. In general, profits and output prices command more labor than the total value output, so that neither of Marx's conservation laws 2 or 3 are satisfied.

$$p_{1w}y_1(t) + p_{2w}y_2(t) = \frac{p_1}{p_2}(e+1)y_1(t) + (1+e)y_2(t) \approx 1.68y_1(t) + 2y_2(t) \tag{92}$$

We can see immediately that this will not equal the total output value, and in fact is nearly double this. What is going on here? The answer is exploitation. The worker, in our concrete model, supplies double the hours of labor than what the capitalist pays for. However, the products that the worker produces during this 'free' time still have a price. Overall prices, therefore when seen as expressing an amount of potential command over labor, will always express more labor than the total overall value. This is not because more value was conjured out of thin air, but simply because *in a society with exploitation, labor-power itself is fundamentally undervalued.*

Why is the total price out a bit less than double the total value output? This has to do with the correction required to facilitate equalized profits. Department I has the lower composition of capital, with $k_1 = 1$ and $k_2 = 4$. To correct the differences in profits, the actual price of commodity 1 must be underpriced, relative to it's value. This underpricing means less labor commanded per unit than the rate of exploitation would dictate. Despite these issues, we can see that Marx was fundamentally correct to assert the overall fact: surplus value is redistributed in a manner whcih is fair to all capitalists, despite the technical differences in production between sectors.

Turning to total surplus value vs total profit, we know already that total surplus is $s_1y_1(t) + s_2y_2(t)$. For total profit, note that the profit earned from producing a single unit of commodity 1 is $p_1 - (p1 * c1 + p2 * v1)$, i.e. revenue minus cost. Since this unit is an hour of value, we can simply multiply this number by $y_1(t)$ to get the total profit in department 1. Finally, dividing by the wage w gives this amount in hours of labor commanded. The same can be said for department 2, so that total profit is defined

$$(p_{1w} - (p_{1w}c_1 + p_{2w}v_1))y_1(t) + (p_{2w} - (p_{1w}c_2 + p_{2w}v_2))y_2(t) \approx 0.45y_1(t) + 0.54y_2(t) \tag{93}$$

Here we see that department 2 earning more absolute profit more per hour of output value, and this is simply because it has a higher operating cost. The absolute profits might be higher, but the profit rate is the same. We can also see that both of these numbers are higher than what the total surplus value would be, and this is for the same reasons that we noted earlier: exploitation effectively inflates all observed price outcomes above their values. Figure 17 shows plots of total prices vs total value and total profits vs total surplus values respectively.

Morishima's conclusion from this is both of Marx's conservation laws 2 and 3 are incorrect. The fact that total surplus value does not equal total profit in particular seems at first like a major issue for Marx, as it would seem to suggest the existence of a source of profit which is independent of surplus value. This would be a mistake, however. That total profits command more labor than the total surplus value in no way means that additional profit was created out of thin air. Rather, profit commands more labor because when the possessor of profit uses it to purchase labor, they receive more labor in value than they actually pay for it in price. Marx made a mistake in his accounting here, but correcting this mistake in no way undermines his overall theory of exploitation as the hidden driver of profit.[26]

In more recent years, political economists such as Foley, Mosely, and many others have proposed an alternative conversion scheme for normalizing dollar prices as amounts of time. They propose defining the monetary expression of labor time, or MELT, as the ratio of the price of the net product to the value of the net product. The interpretation of this, as we noted, is that the MELT represents at a given moment how many dollars an hour of value is worth. Effectively, the MELT forces dollars to take the place of time unit: dividing a commodity price by the MELT gives then the approximate value of the commodity, just measured in dollars. This is a far superior way of converting prices into amounts of time, as they do not have the exploitation-ignoring inflationary effect that we have seen when converting into amounts of labor commanded. Moreover, that the total price of the net product equals the total value of the net product times the MELT is true by definition of the MELT itself. Thus their MELT confirms that Marx's conservation law 2 can in fact be seen as valid. Our conclusion is that Morishima's assessment of this law is faulty.

The third law, that total surplus value equals total profit, however, remains unresolved by the MELT technology provided by the New Interpretationists. That said, what these theorists propose to resolve this issue is rather a new interpretation of the whole theory of exploitation. Their proposal amounts to seeing the total variable capital not as the direct value of the means of subsistence as Marx proposes in volume I, but rather as an amount which is *post-ante* determined by taking the total wages paid to all workers and converting it into an amount of time by dividing by the MELT. This new definition of variable capital in turn produces a new definition of total surplus value: taking the hourly wage and dividing it by the MELT gives an amount of time which is analogous to the quantity v_1 or v_2. Subtracting this from a full hour thus gives a different interpretation of the amount of surplus time which goes to the capitalist. While we omit demonstrating this from the present paper, Foley et. al. have shown that this redefinition of the theory of exploitation does succeed in upholding Marx's conservation law relating total profit to total surplus value.

However, the first conservation law, from which Marx derived the other two, remains untrue in general. Some theorests, such as Moseley[14], Wolff[17], Kliman[6] and others, advocate for a redefining of the total constant capital via the MELT in the same way that the so-called New Interpretation theorists redefine the total variable capital. One can show that doing this indeed resolves the disparity between the 'value' rate of profit and the equilibrium rate of profit. However, at this point we must acknowledge that we have tampered with the value itself, hence why I put 'value' in quotes in the previous sentence. While different theories of exploitation can be grafted onto the value system without fundamentally changing it, constant capital is different. Constant capital is dead labor. To determine the value of this dead labor *post-ante* via the total money spent on means of production converted via the MELT, we are allowing for price, and thus market and otherwise social forces, both subjective and objective, to feed back into and influence what was once a purely objective and technical measurement. Thus within the systems proposed by these theorists, value is no longer socially necessary labor time, but is rather merely the 'social substance' - a distorted reflection of these labor times as they are continually *perceived* and reevaluated by market forces. This way of thinking about value is pretty neat, especially in the context of volume III, and to the extent that it approximates the actual value system can be operationalized for valuable empirical testing.

[26] Ian Wright provides a more mature explanation for these observed disparities[19][18]. In his PhD thesis, he demonstrates that there is a way to generalize the labor theory of value in such a way that it includes both the labor required to reproduce the capitalist class as well as the labor required for growth as 'necessary labor'. Under such a valuation system one finds that these so-called 'super-integrated' labor values *are* directly proportional to equilibrium prices, with constant of proportionality the hourly wage rate. This means that the equilibrium prices of production, measured in hours of labor commanded and obtained by dividing by the unit wage, are exactly equal to his super-integrated labor values. In a sense, we can take this as seeing things more from the perspective of the capitalist class, who would argue to us that the inflated labor commanded by prices are necessary, for otherwise there would be no other way for them to exist or for the economy to grow. This is a brilliant discovery, but in retrospect it is also no great surprise that the definition of labor value which perfectly coincides with natural prices is the one which actually includes the extra labor created for a 'leeching' class alongside the actually necessary during the production of commodities as. necessary.

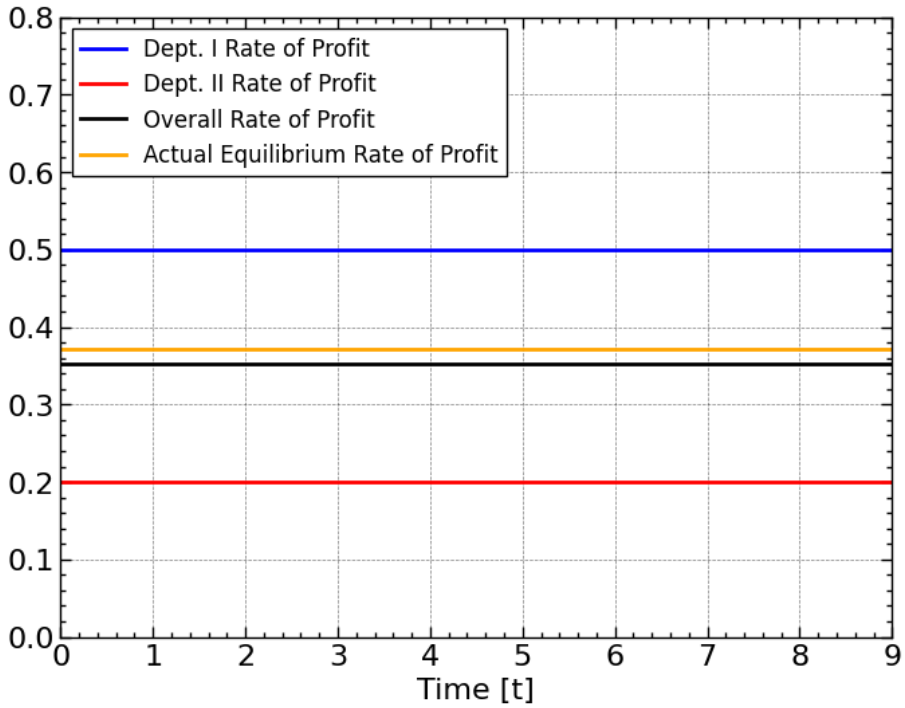

Figure 18: The problem with Marx's transformation problem. The equilibrium rate of profit experienced by capitalists is not generally equal to the value rate of profit, even on a balanced growth path.

However, one can only see these as solutions to the transformation problem, partial or otherwise, if one is only interested in 'redeeming Marx'. I would strongly argue against thinking this way. The transformation problem should not be seen as a controversy about whether or not Marx was 'correct about everything'. It is, at its core, an academic discussion about the extent to which the technological parameters of a capitalist society constrain its price structure. The revised notion of what is meant by 'value' that these theorists suggest may indeed have some economic or otherwise philosophical significance, but they are completely irrelevant to that aforementioned discussion, because they are no longer purely related to the techniques of production. A true solution to

It solves the problem by simply turning down the focus of our analytical lens. In other words, to the extent that this is a solution to the transformation, it only 'solves' the problem by avoiding it.

7.0.3 Equilibrium Profit Rate vs Value Profit Rate: Unequal, But Related

Figure 16 shows the the overall value rate of profit (along a balanced growth path) along with the equilibrium money rate of profit experienced by capitalists. It also displays the two departmental rates of profit. Just as was the case with total prices and total profits, the equilibrium money rate of profit is generally seen to be *higher* than overall value rate of profit along the balanced growth path. However, the reason I have chosen to include the departmental rates of profit is to demonstrate that despite this difference, *both* rates of profit remain bound between the lower and higher of these departmental rates, and this remains true no matter what our system's parameters are.

This boundedness of the equilibrium rate of profit has a number of extremely important implications. First, in a broad sense, it means that the claims which Marx makes about his value rate of profit throughout the volumes of capital apply *residually* to the equilibrium rate of profit, despite them not being equal. This includes his argument for a technologically induced falling rate of profit. For the uninitiated, note that the departmental rates of profit can be written:

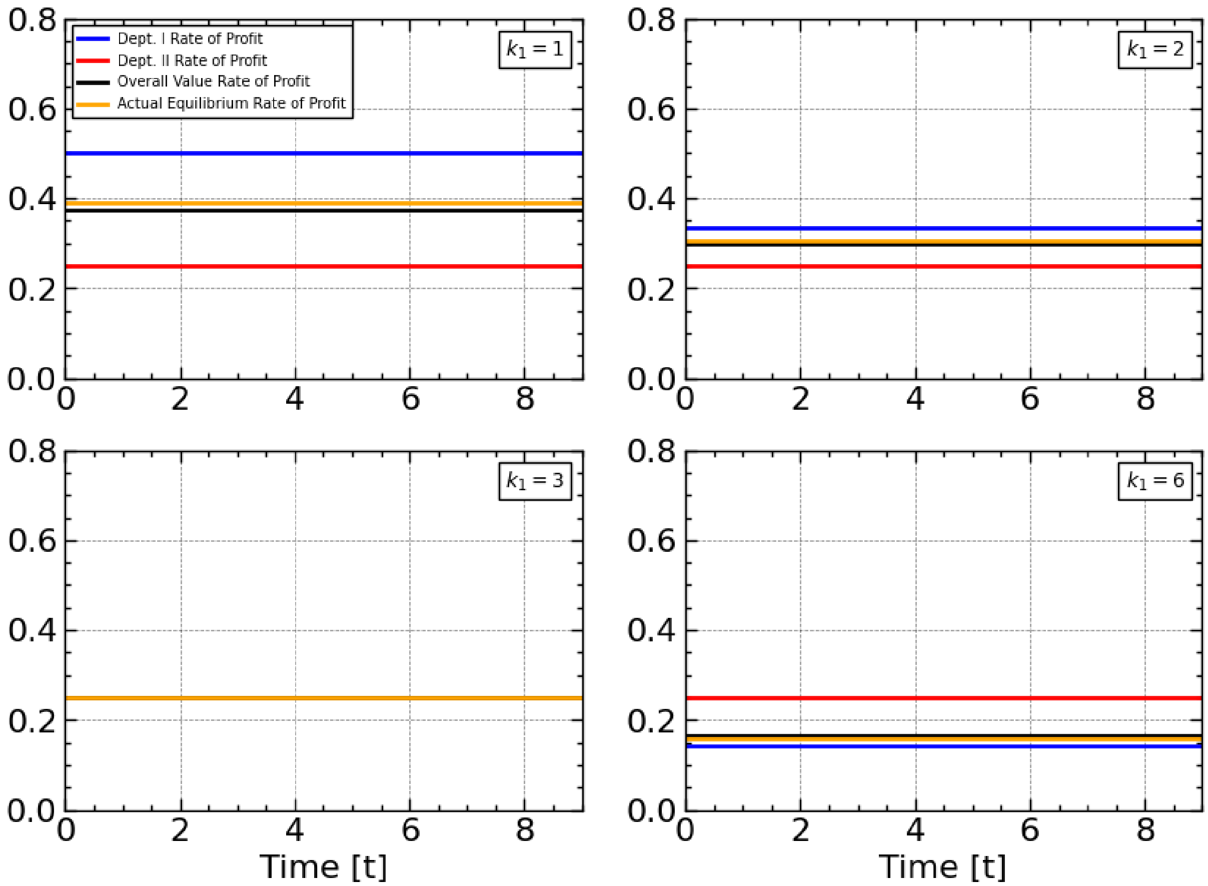

Figure 19: Effects of the changing departmental compositions of capital on both Marx's value rate of profit and on the equilibrium money rate of profit actually experienced by capitalists. Other relevant parameters here are that $k_2 = 3, e = 1$. As k_1 increases, the rate of profit for department I falls, pushing down *both* rates of profit at once. When $k_1 = 3 = k_2$, departmental profit rates equal one another, forcing the value rate of profit to equal the equilibrium rate of profit. When we continue increasing k_1 such that dept. I's profit rate is now lower than dept. II's, we see that lowering the smaller of the two departmental rates *pulls* both profit rates down with it.

$$\pi_i = \frac{s_i}{c_i + v_i} \frac{\frac{1}{v_i}}{\frac{1}{v_i}} = \frac{\frac{s_i}{v_i}}{\frac{c_i}{v_i} + 1} = \frac{e}{k_i + 1} \tag{94}$$

With e held constant, an increase in k_i amounts to an increase in the denominator, and thus a decrease in π_i. In section 6.6 we discussed Marx's argument for a growing composition of capital over time. We can now see that what follows from that is his (in)famous argument for a tendency of the rate of profit to fall over time: increases in k_1 or k_2 induce an increase in $K(t)$, which in turn induces an increase in the overall value rate of profit π. The fact that $\pi \neq \pi_e$ would seem to throw a wrench in this argument. However, the boundedness of π_e between the two departmental rates of profit allows us to observe that this equilibrium rate of profit will still feel the effects of these technological changes. The reader is encouraged to adjust the k_i sliders themselves and see an increase to the k_i corresponding to the more profitable department pushes down the corresponding departmental profit rate, the equilibrium profit rate and the overall value profit rates simultaneously. Likewise, increasing the k_i for the less profitable sector *pulls* down the three respective rates. Figure 19 shows this.

Okishio's theorem is largely seen as a general refutation of Marx's argument for a technologically induced falling rate of profit. However, Okishio's theorem notably does not hold the rate of exploitation constant, as our model does. Okishio's model implicitly allows real wages to float, so that the value of labor power continually falls as labor saving technology is introduced. Marx's argument was not simply that the rate of profit would fall as technological innovation progressed under capitalism. Rather, it argues that the rate of profit will fall due to this technological progress *unless* exploitation increases to compensate. Roemer showed that under general conditions, if the real wage is adjusted so that the wage share remains constant before and after cost-reducing technological changes are introduced, then the equilibrium rate of profit falls[15]. Our model demonstrates the same principal, not by holding the wage share constant but rather by holding the rate of exploitation constant. The principal is this: cost-reducing and super-profit generating technological changes, once generalized, tend to increase the rate of profit *as well as* the rate of exploitation simultaneously. However, that rise in the rate of profit can quickly invert itself into a fall in the rate of profit *if* the working class, through struggle, manages to share the benefits of that increase in labor productivity, to any degree.

Moving on, we can verify that despite the danger seemingly introduced by the equilibrium rate of profit not equaling the overall value rate of profit, *it remains the case that exploitation is a necessary condition for profit.* Note that for either department, continuing from where we left off in equation 94:

$$\pi_i = \frac{e}{k_i + 1} \leq e \tag{95}$$

Since the overall value rate of profit is bounded below the larger of these departmental profit rates, it follows that Marx's value rate of profit is bounded below the rate of exploitation. This is the essence of Marx's argument for why exploitation is a necessary condition for profit in a capitalist system: if $\pi \leq e$, and $e = 0$, then it follows that $\pi = 0$. Thus, no exploitation means no profit.

Once again, it would seem at first glance that the inequality between the equilibrium rate of profit and Marx's value rate of profit throws a wrench in this argument, compromising one of the central pillars of Marxist thinking. However, we can easily see that this is not the case, because the equilibrium rate of profit is *also* bounded below the larger of these departmental profit rates. The reader is encouraged to adjust the e slider towards 0 and see for themselves that both profit rates fall in turn, finally meeting and equaling 0 when $e = 0$. This fact, that the rate of exploitation remains bounded above the rate of profit, even in an economy with equalized profit rates, is known as the *fundamental Marxian theorem.* This theorem marks a milestone in the historical resilience of Marxist theory in the face of everything that has been thrown at it.

To summarize our findings, we say first and foremost that the controversies surrounding the transformation problem are entirely independent of the controversies pertaining to a technologically induced falling rate of profit. Despite the equilibrium rate of profit not equaliing Marx's overall value rate of profit, both obey the same dynamics with respect to a changing overall composition of capital and the rate of exploitation.

Many Marxists since the controversies about the transformation problem sought to escape them by criticizing the equilibrium models themselves. While some of these alternative models are quite sophisticated and have value in and of themselves, this approach to the transformation problem is fundamentally one of escapism. While the equilibrium model is problematic for reasons we have already addressed, it's asymptotic simplicity allows us to witness certain fundamental truths of the capitalist system which would otherwise be impossible to see. If we accept the transformation problem as real, there are new and fascinating insights to gain about how money works in our economy. Let us now turn to one such observation.

7.1 Reinventing the Model

In lieu of this knowledge that the prices and profits are different from values and surplus values in our equilibrium economy, our reinvestment schema is called into question. Recall that we had capitalists reinvesting a fixed proportion of their *surplus value* each period. This is no longer a realistic assumption, because capitalists are not even aware of the surplus value they are receiving. A more reasonable reinvestment scheme now would be to assume capitalists are reinvesting fixed proportions of their *profit.*

In his book, Morishima reformulates the entire system under this more realistic assumption.[27] This is not terribly hard to do now that we've worked out how to think about prices within our system. Despite

[27] He calls this task of transforming the rate of surplus value reinvestment into a rate of profit reinvestment the *dynamic transformation problem.* This is a fun framing, but not very consequential in my opinion.

capitalists reinvesting based on prices, we are still operating within our labor time accounting system: $y_1(t)$ and $y_2(t)$, as before, represent the total combined labor value of all commodities available for purchase at time t. Our system of equations will look almost identical to before; the real physical requirements of our system do not change. The only thing that we need to re-evaluate is how many hours worth of value capitalists are expected to consume each period. Before, this was simply $(1-a)(s_1y_1(t)+s_2y_2(t))$, i.e. the remainder of whatever surplus they did not reinvest.

As we did earlier, we are assuming for simplicity that the value of a unit of output from each department is 1 (i.e. one 'unit' of output is simply an hour of value). Focusing without loss of generality on department I's output value $y_1(t)$, then the value of the means of production needed for producing this is $c_1y_1(t)$, and likewise the value of the labor goods needed to pay workers is $v_1y_1(t)$. By our tactical conflation of hours of value with real units, this is also the number of units of each commodity type which is needed. Thus we can multiply each of these numbers by the price to calculate the price of these. The total price of the capital investment required to produce the output value $y_1(t)$ is therefore $(p_1c_1+p_2v_1)y_1(t)$.

Now, since our system assumes a uniform rate of profit π_e, we have that the total price of the output from department I is $(1+\pi_e)(p_1c_1+p_2v_1)y_1(t)$, and so the portion $\pi_e(p_1c_1+p_2v_2)y_2(t)$ is the total profit capitalists from extract from the department I. Of this, they reinvest the proportion a of it, leaving the dollar amount $(1-a)\pi_e(p_1c_1+p_2v_1)y_1(t)$ for consumption. The same process repeated for department II gives a combined capitalist consumption fund of

$$(1-a)\pi_e((p_1c_1+p_2v_1)y_1(t)+(p_1c_2+p_2v_2)y_2(t)) \text{ dollars} \tag{96}$$

Now we need to convert this amount back to value terms. Since there is only one good which is suitable for consumption, we can divide by its price to get the number of units of that good which this dollar amount will purchase. However, once again by our tactical conflation of real units with units of output, this number is also precisely the total value of goods consumed by capitalists. We have found our replacement for the value of goods consumed by capitalists:

$$(1-a)\pi_e\left(\left(\frac{p_1}{p_2}c_1+v_1\right)y_1(t)+\left(\frac{p_1}{p_2}c_2+v_2\right)y_2(t)\right) \tag{97}$$

With this drop in replacement ready, the new system of difference equations can be stated:

$$y_1(t)=c_1y_1(t+1)+c_2y_2(t+1) \tag{98}$$

$$y_2(t)=v_1y_1(t+1)+v_2y_2(t+1)+(1-a)\pi_e\left(\left(\frac{p_1}{p_2}c_1+v_1\right)y_1(t)+\left(\frac{p_1}{p_2}c_2+v_2\right)y_2(t)\right) \tag{99}$$

Our system makes the same statements as it did before. The first equation says that the means of production used in the production of the output for time $t+1$ is exactly what is available in means of production at time t. The second says that the value of consumer goods consumed by workers and capitalists between periods t and $t+1$ is exactly the value in such goods which is available at time t. The system is structurally identical to the original system 38, just with some complication introduced in expressing the value of goods consumed by capitalists, since it no longer relates as directly to the output values. Because of this, the resulting model looks and behaves nearly identically to the original model. We see the same disproportionality crises, the same chaotic hiring and firing practices, and so on. There is just one crucial and fascinating difference.

Figure 20 shows what the paths for our numerical example of 5 looks like. These paths can be found under the 'growth paths related to the system' tab underneath the growth paths we have been looking at and are denoted y_{1p} and y_{2p} dual quantities for all relevant variables can be found throughout the desmos app, and one can tell they are related to this new system by the p subscript. We see similar overall behavior to what has already been discussed except with slightly different numbers. Everything we have stated so far still applies, including the fact that the behavior of the system changes fundamentally when $k_1 > k_2$.

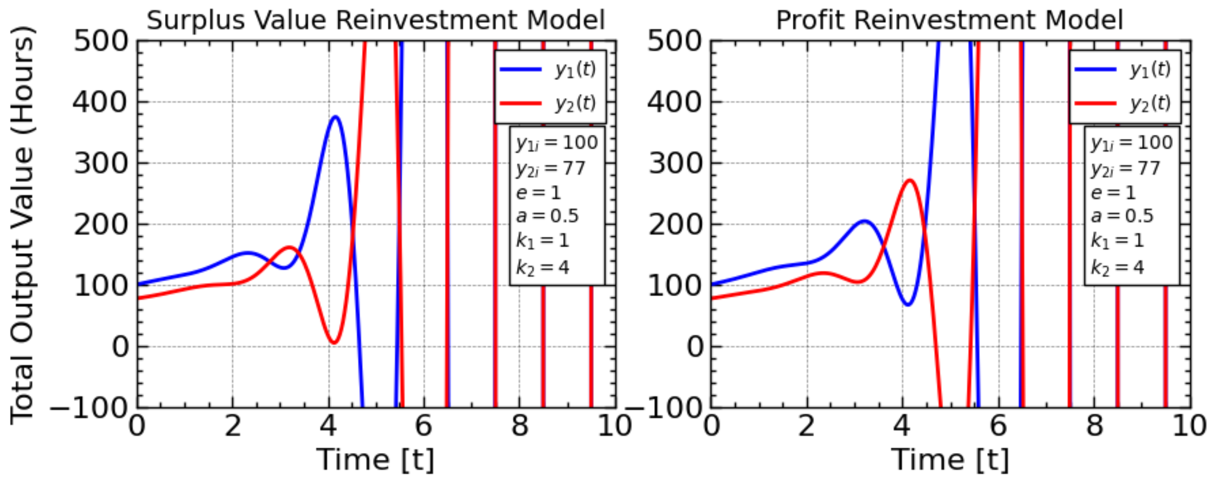

Figure 20: Comparison of the warranted growth paths between the two models with identical system parameters. While the exact curves differ, the qualitative results are identical (with one exception to be discussed).

7.2 Stability and Information

It is only within this more realistic model that money truly takes on the role of obfuscating agent which Marx saw in it. Capitalism is a system. All systems seek to maintain stability of some sort.[28] The question of *what*, exactly, a system is stabilizing, is of the utmost importance, because stabilization of one signal always comes at the cost of instability elsewhere. Effective analysis of any system should thus endeavor to quantify what, exactly, a system is seeking to stabilize, and what other forms of stability are being ignoreed in this process.

In an information theoretical sense, systems maintain stability by *absorbing*, or *filtering* incoming information[29]. The classical example of this is the thermostat. The thermostat is a control system which intends to maintain a stable temperature. Changes to the outside temperature bleed into the system, and are detected by the thermostat, which takes counterveiling action to prevent the inside temperature from being effected by this. The result of the successful action of the thermostat is the destruction of information. We know the thermostat is working when we *do not notice these external changes*.

While the stabilizing of a system can always be seen conceptually as the system absorbing information, it would be a grave mistake to associate, in general, the absorption of information with the stability of a system. The reason for this discrepancy comes from the dangers introduced by measurement. The systems we care about are real, and the stabiltiy we hope to achieve is usually qualitative. In both the modelling of real systems and in the implementation of control systems in real life, we must substitute for this qualitative stability a quantitative measure - a metric. Stability of the system is then associated with this number remaining the same over time. However, the metric almost never correlates one-to-one with the real stability we care about. For example, consider again the example of the thermostat. The stability we are hoping to maintain in this example is *not* temperature, but rather our *comfort*. Temperature is merely the numerical metric which we substitute for comfort in the design of the thermostat. There are a number of dimensions to our comfort which the temperature metric ignores, a solid example being the humidity of the room. As a result, we tend to have rooms during winter which are kept successfully at a comfortable temperature, but are nonetheless uncomfortably dry.

It is therefore quite easy to create systems which *appear* stable, but which nonetheless produce catastrophic results in unintended ways. Information absorption, *as witnessed by the metric*, need not be the

[28] A system, at the very least, is something which is capabable of existing for some period of time. This entails the maintenance of a stable distinction between the system itself and it's surrounding environment. The continual process of self-maintenance, or self-production, was originally identified by the biologists Maturana and Varela as *autopoeisis*[16]. Stafford Beer pioneered the adoption of this valuable concept to the cybernetic analysis of social systems.[1]

[29] I am of the opinion that the concept of variety as per Ashby and Beer is mathematically identical to Shannon information (i.e. statistical entropy). I choose to use the two terms interchangeably throughout this paper.

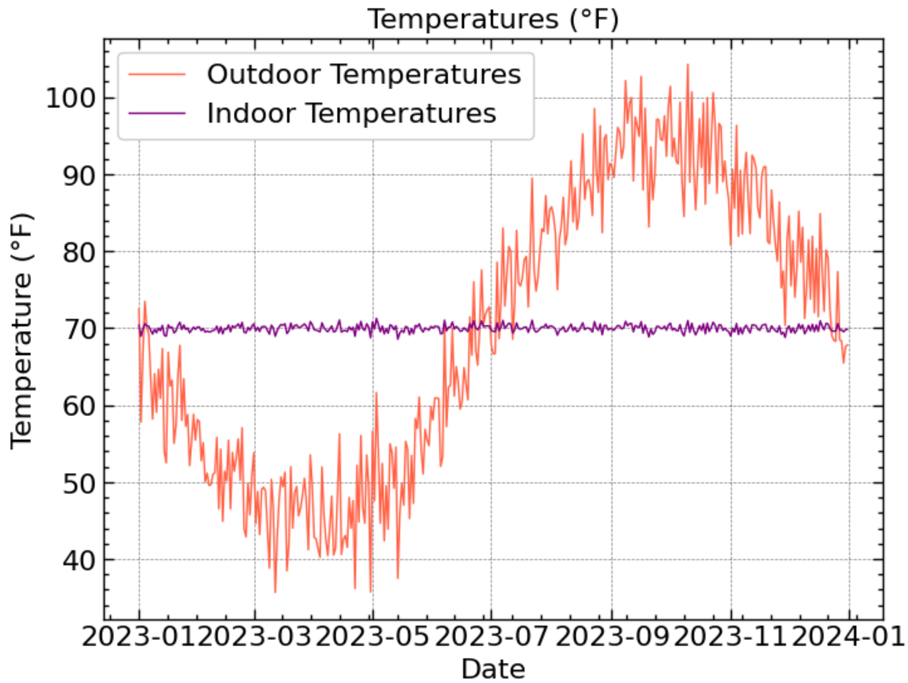

Figure 21: A thermostat maintains a stable indoor temperature. To the degree that it does so successfully, it destroys information about the outdoor temperature.

result of the control system taking action and providing the quantitative stability which desired. It can also due to the *metric* itself, with no action taken by the system. Most importantly, the reader should not assume that this always takes place in the benign sense of the metric simply not taking into account a certain dimension of the desired stability (e.g. humidity in addition to temperature). It can also happen that the particularities of the system cause the metric to take on the unintented role of an *anti-metric*, which *actively* filters out information in unintended or unintuitive ways. To see an illustration of this, let us return to our model. Seeing it as a system, let us try to identify what is stabilized, what is unstable, and what money-based signals might be actively hiding from the capitalists overseeing it.

We have already identified several properties of our model which are notably *unstable*. In particular, we have identified that stable and desirable outcomes in terms of the output of use-values is something which our system does not seem at all concerned with. As a residual side-effect, it seems equally unconcerned with stability in the demand for labor. However, what we are about to see is a peculiar way in which money signals serve to actively *hide* this instability from any actors who might be monitoring it.

7.3 Money: The Mask of Capitalism

What is stabilized within our toy equilibrium economy? One obvious answer is prices. However, while this is true, price stability is only characteristic of an economy maintaining equilibrium. Our model is defined by the maintenance not just of equilibrium, but also of steady growth. We seek to identify a metric whose stability corresponds with the growth aspect of our system. While we have defined *how* the system grows, we have yet to try and identify *what* exactly is growing. My answer to this question (and I do not foresee any disagreement here from fellow Marxists) is capital. Capital accumulation is the growth imperative of our system.

Informally, we can define the rate of capital accumulation as the proportional change in total capital investment between times t and $t + 1$. This is informal, because we have two ways of measuring the total capital investment. We can choose to measure this total investment in terms of money, or in terms of price. It should be noted that such a choice does not exist unless we acknowledge and come to terms with the transformation problem. Those Marxists who attempt to escape the controversy by coming up with alternative theories which 'solve' the transformation problem by simply smothering the distinction between prices and values do so at the potential cost of the descriptive potential of their alternative theories.

If we choose to measure the rate of accumulation in terms of value, then then portion of the total output value which constituted capital investment during the prior production process is $(c_1+v_1)y_1(t)+(c_2+v_2)y_2(t)$. Thus the rate of capital accumulation, measured in terms of growth the value of total investment, is

$$G_v(t) = \frac{(c_1+v_1)\Delta y_1(t) + (c_2+v_2)\Delta y_2(t)}{(c_1+v_1)y_1(t) + (c_2+v_2)y_2(t)} \tag{100}$$

On the other hand, the constant capital investment $c_1 y_1(t)$ has a dollar price $p_1 c_1 y_1(t)$, and the variable capital likewise has price $p_2 v_1 y_1(t)$. Thus the dollar *price* of the capital investment for producing $y_1(t)$ is $(p_1c_1+p_2v_1)y_1(t)$. Similarly the dollar price of the capital investment for producing $y_2(t)$ is $(p_1c_2+p_2v_2)y_2(t)$. Thus the rate of capital accumulation, measured in terms of growth in the price of total investment, is

$$G_m(t) = \frac{(p_1c_1+p_2v_1)\Delta y_1(t) + (p_1c_2+p_2v_2)\Delta y_1(t)}{(p_1c_1+p_2v_1)y_1(t) + (p_1v_2+p_2v_2)y_2(t)} \tag{101}$$

Either of these metrics could be taken as a measure of economic 'health' with respect to our system's growth imperative, in the sense that we could (and the capitalist class likely would) interpret either number remaining stable at a particular value as a sign that the system is behaving in an ideal way. However, we can observe that one of these metrics properly measures the stability of an economy in terms of use-values, while the other does not.

Let us begin our analysis of these growth metrics with the value rate of accumulation $G_v(t)$ in the original model where capitalists are reinvesting a fixed proportion a of their surplus value each period. Under this assumption, any change in the total value of the capital investments must be entirely attributed to that proportion of the total surplus, i.e. $(c_1+v_1)\Delta y_1(t) + (c_2+v_2)\Delta y_2(t) = as_1y_1(t) + as_2y_2(t)$[30] But then if we plug this back into equation 100, we see

$$\begin{aligned} G_v(t) = \frac{(c_1+v_1)\Delta y_1(t) + (c_2+v_2)\Delta y_2(t)}{(c_1+v_1)y_1(t) + (c_2+v_2)y_2(t)} &= \frac{as_1y_1(t) + as_2y_2(t)}{(c_1+v_1)y_1(t) + (c_2+v_2)y_2(t)} && (105) \\ &= a\frac{s_1y_1(t) + s_2y_2(t)}{(c_1+v_1)y_1(t) + (c_2+v_2)y_2(t)} && (106) \\ &= a\pi(t) && (107) \end{aligned}$$

I.e. the rate of capital accumulation in value terms is simply a times the value rate of profit. However, we already discussed in section 7.0.1 how the rate of profit changes according to disproportions between departments. If, for example, department I has the higher composition of capital, and our society begins to massively divest from this department, then society will in turn become more labor intensive, leading to a spike in $\pi(t)$. We can see then that this metric is not at all stabilized by our system - it changes to reflect disproportions in use-value as they arise. We should see this as a point in *favor* of the metric, however. The system itself is what is unstable - the metric is only reflecting this instability in an honest manner. If we, as overseers of such a system, were monitoring the rate of accumulation measured in value terms, we would see disproportions arising through it, and perhaps be able to act in order to stabilize the system.

The money rate of accumulation $G_m(t)$ behaves similarly in this model, and this is fundamentally because capitalists are reinvesting their surplus value and *not* their profit. If, for example, our society shifts to

[30] To see this algebraically, if we begin with our difference equations 38. Distributing across the two $1-a$ terms and adding these two equations together and rearranging gives

$$(c_1+v_1)y_1(t+1) + (c_2+v_2)y_2(t+1) = y_1(t) + y_2(t) + s_1y_1(t) + s_2y_2(t) - as_1y_1(t) - as_2y_2(t) \tag{102}$$

Substracting the two $y_i(t)$ terms over to the left side then and factoring then gives

$$\begin{aligned} &(c_1+v_1)(y_1(t+1) - y_1(t)) + (c_2+v_2)(y_2(t+1) - y_2(t)) = a(s_1y_1(t) + as_2y_2(t)) && (103) \\ &\Leftrightarrow (c_1+v_1)\Delta y_1(t) + (c_2+v_2)\Delta y_2(t) = as_1y_1(t) + as_2y_2(t) && (104) \end{aligned}$$

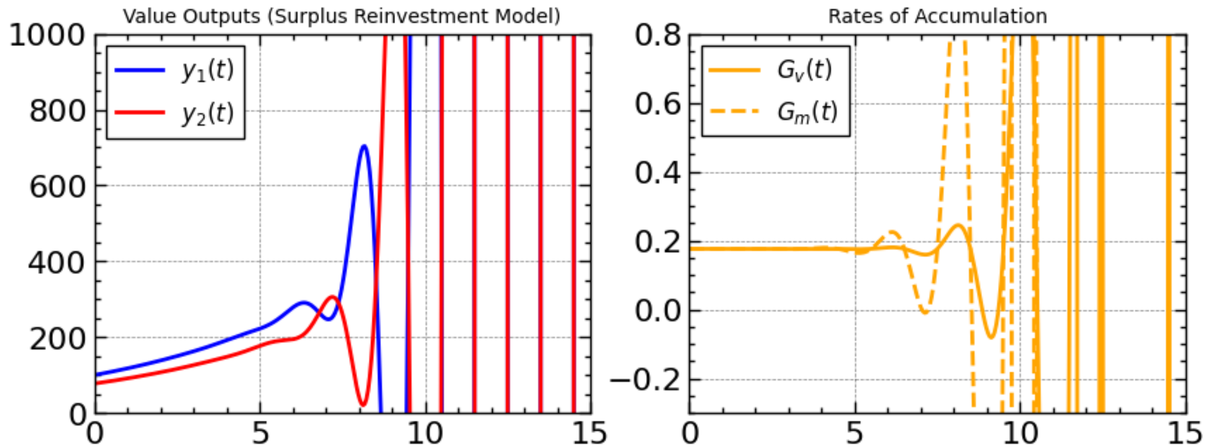

Figure 22: In the original version of our model, the rate of accumulation accurately reflects use-value instability whether regardless of whether the unit of measurement is value or price.

overemphasizing the less labor intensive department, then this will result in a sudden fall in the total surplus value which capitalists are extracting, leaving less for reinvestment and a slowdown in capital accumulation both in terms of value and price. Figure 7.3 shows both how both $G_v(t)$ and $G_m(t)$ look in our original surplus-reinvestment model.

Let us now move on to our new model of profit reinvestment, though it is harder to observe analytically, it remains the rate of accumulation measured in terms of value accurately reflects changes and emergent instabilities in the proportional output of use-values. This is fundamentally for the same reason as before, though it is harder to show analytically here. If, for example, society begins to suddenly overemphasize the more capital intensive department, then this would result in a reduction in the surplus value available in the next period, and this would correspond to a reduction in the overall growth of the economy in value terms, regardless of how the capitalists choose to reinvest.[31]

Now, however, we finally arrive at the interesting case: the rate of accumulation in money terms, in the model where capitalists are reinvesting a fixed proportion of their profit each period. It should be noted that this is the most realistic situation to consider of the four, since now we are entirely operating at the level of a capitalist in the 'real world', operating entirely on money signals, and entirely unaware of the world of values. Here the assumption of a uniformally equalized profit rate has its effect on the money signal's ability to accurately measure the stability of the economy it supposedly regulates. We noted in the case of the surplus-reinvestment model that the change in value of the capital invested between times t and $t+1$ must be entirely attributable to the surplus value which is reinvested. We can say something identical here about the change in price of the capital invested between times t and $t+1$: it must be entirely attributable to the fixed proportion of the profit which was reinvested. In other words, we can say

$$(p_1c_1 + p_2v_1)\Delta y_1(t) + (p_1c_2 + p_2v_1)\Delta y_2(t) = a\pi_e((p_1c_1 + p_2v_1)y_1(t) + (p_1c_2 + p_2v_2)y_2(t)) \tag{108}$$

But then if we plug this into the numerator of equation 101, we see

$$G_m(t) = \frac{(p_1c_1 + p_2v_1)\Delta y_1(t) + (p_1c_2 + p_2v_2)\Delta y_1(t)}{(p_1c_1 + p_2v_1)y_1(t) + (p_1v_2 + p_2v_2)y_2(t)} \tag{109}$$

$$= a\pi_e \frac{(p_1c_1 + p_2v_1)y_1(t) + (p_1c_2 + p_2v_2)y_2(t)}{(p_1c_1 + p_2v_1)y_1(t) + (p_1v_2 + p_2v_2)y_2(t)} \tag{110}$$

$$= a\pi_e \tag{111}$$

[31]This is only mostly true. If the capitalists suddenly chose to massively ramp up their reinvestment rates, then this could account for the absolute reduction in surplus value income. But the fixed reinvestment rate, either in value or in profit, restricts their ability to do this too severely for it to matter here.

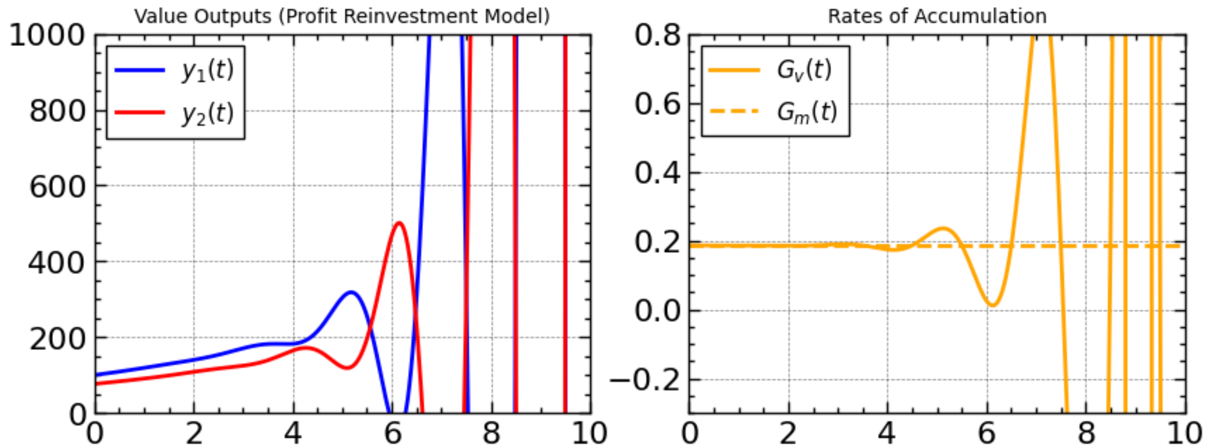

Figure 23: In the revised model where capitalists reinvest a fixed proportion of their profits, they are guaranteed equal returns on their investments regardless of the use-values produced. Thus the rate of accumulation measured in money fails to accurately reflect the stability of the economy.

The nature of the cancellation is similar to what we saw with the value rate of accumulation in the first model, but now instead of the overall value rate of profit $\pi(t)$ we instead see the π_e, which *unlike* $\pi(t)$ *is constant with time.* Thus the signal becomes perfectly stable, *in spite of any disproportions in use-values.*

Why does this happen? The key is understanding that this is an implication of uniform rates of profit. A uniform equilibrium rate of profit means that capitalists receive the same return on investment *regardless* of the actual industry they are investing in. This means that all use-values appear equally profitable. Since all investments are equally profitable regardless, the profit income will remain steady regardless of which department is being emphasized. As long as the profit income remains steady and as long as we are assuming a fixed proportional reinvestment of those profit, the result is a rate of accumulation which itself remains steady. Thus we have reached a startling result. In an equilibrium economy, money perfectly takes on the role of *anti-metric.* Any metrics viewed in terms of money will smother out all information about the world of use-values. Thus in an equilibrium economy, *the world of exchange-value finds itself entirely divorced of the world of use-value.*

Morishima himself seems excited by this result, but does not stop to think about what it really means. From a cybernetics standpoint, this result is a fascinating example of metrics serving to actively destroy information not pertaining to the exact form of stability being measured (more on this in a moment). From a Marxist theory standpoint, this result is perfectly in parallel with all of Marx's observations about money, and more broadly the contradiction between use-value and value, that he makes throughout the volumes of Capital. Not only do markets fail to adequately produce desirable results for it's participants in terms of use-values, the 'dazzling money-form' blinds capitalists to this reality. If they only measure the health of the economy through the lens of their pocket-books, they will sleepwalk society off of a cliff. Complexities of the structure of money signals facilitate this obfuscation. Money is therefore an agent of obfuscation, just as Marx sought to show in his attempt at solving the transformation problem.

While Marx's solution to the transformation problem is deeply flawed, if we acknowledge these flaws and attempt to grapple with where the flaws originate, we find a hidden witness which directly serves in the interest of Marx's goals which led him to approaching that problem in the first place. Moreover, if we look more closely at the issues, we see that Marx's overall theory remains unscathed. Without exploitation, there is no profit, even after we accept that the rate of surplus value is not equal to the rate of profit.

We saw earlier in this section that Marx's value rate of profit is changing with time, violently oscillating in tandem with the swings in the value output of the departments. Capitalists in the real world are very concerned with the rate of growth of the economy. If the rate of accumulation in price terms demonstrated the same behavior as this, then they would be forced to admit that something is wrong with the price signal. Incredibly, when you measure the rate of capital accumulation in price terms, this information completely disappears. Figure 7.3 shows the graphs of both the rate of accumulation in value terms (the solid orange line, $G(t)$ in desmos) and the rate of accumulation in price terms (the dotted orange line, $G_p(t)$ in desmos). plotted against each other. From this we see that even in the midst of a real crisis of disproportionality, the capitalists, if they are assessing the economy entirely on price signals, will *completely fail to recognize what is happening.*

8 Conclusions

A great deal of justified criticism has been levelled at the so-called neo-Ricardian equilibrium modelling tradition in recent years. The unrealistic premise makes it difficult and dangerous to interpret the dynamics obtained within such models directly onto the real world. Thus we conclude the paper with a discussion of how these observations might be applied to responsibly derive new 'laws of motion of the capitalist system'.

It must be admitted that Marx ended his life convinced that the cause of all real crises in capitalism is what we called the crisis of overproduction/underconsumption: "...a crisis would be explicable only in terms of a disproportion in production between different branches and a disproportion between the consumption of the capitalists themselves and their accumulation. But as things actually are, the replacement of the capitals invested in production depends to a large extent on the consumption capacity of the non-productive classes; while the consumption capacity of the workers is restricted partly by the laws governing wages and partly by the fact that they are employed only as long as they can be employed at a profit for the capitalist class. The ultimate reason for all *real* crises always remains the poverty and restricted consumption of the masses, in the face of the drive of capitalist production to develop the productive forces as if only the absolute consumption capacity of society set a limit to them."([10] p.615, my emphasis) The context preceding this quote is unimportant. What is important is the fact that it shows that Marx acknowledged the theoretical possibility of the crisis of disproportionality, but that he believed the crisis of overproduction/underconsumption took precedence and would be the cause of all of capitalism's crises nonetheless.

Morishima argues based on this quote that Marx would not have accepted either of the models we have proposed here, for both describe a crisis cycle which is not based on the contradiction between the need for increasing worker immiseration and the need to find markets for the commodities of an increasingly productive system. However, it is unclear whether Marx would have still agreed with his claim here if he had completed his objectives in volume II. If we for the current moment take our observed disproportionality crises as manifesting in the real world as 'real' crises, then I believe that they are of a qualitatively different sort from the crisis of overproduction. As we've noted, the crisis of disproportionality can be resolved and deferred to a later date through an intervention which resets departmental outputs to numbers better approximating balanced growth. In contrast, the crisis of overproduction casts a dark and impenetrable cloud over capitalism which cannot simply be resolved overnight without a large scale confrontation between capital and labor.

I thus agree with Marx in his declaration here of primacy of the crisis of overproduction from the standpoint of what does and does not have the potential to actually bring about the end of capitalism. However, I believe that Marxists up until now have been far too fast in attributing the many smaller crises of capitalism - those which rock the system but do not threaten it, to the crises of overproduction, and that this weakens their case for it as an existential threat. Was the crisis of the 1840s which led to revolution all over Europe a crisis of overproduction? I think not. It is widely accepted that the primary causal factors of that crisis were the bursting of the speculative bubble of the railroad industry and the surging of food prices resulting from the potato blight (itself due to capitalism's overreliance on monoculture). The railroads were not overproduced, but rather overinvested in. This has the flavor of the disproportionality crises we have been witnessing theoretically - mass overinvestment in railroads on the one hand can be seen as mass divestment from other industries. Today, we find ourselves facing a similar bubble on the brink of bursting, not from the railroad industry but rather form so-called artificial intelligence. Nobody actually *wants* this new technology to take over. Yet markets act as if we do. Why are markets acting this way? The standard Marxist answer would be to say that it is simply the next iteration of attempting to wage class war on the skilled labor force and automate their jobs away. That would be the Volume I answer. The volume II answer though, which is equally valid, would be simply this: exchange-value growth at all costs. The numbers must increase at a competitive rate.

Next question: can the inevitable bursting of the AI bubble be seen as a disproportionality crisis? To answer that, we need to ask if the investment in AI is at the cost of investment in otherwise more desirable use-value production. One look at the state of the housing market in any country of the so-called first world will confirm that this is certainly the case. One crucial difference to note here is the reason that the housing market is being divested from in favor of other industries. Within our model, the reason is always the same: resource constraints. With the housing market, the reasoning is far more political: those who already own property see the building affordable and high density housing as a threat to their property values, and put up resistance to these efforts. This points to a crucial difference in general between our model's findings and real life - something else *besides* resource constraints usually crop up first. However, observe that all of these other reasons are merely further fuel for critique of the market system. They pile on as further examples of why capitalist production is not at all a function of people's actual desire for use-values. What our model serves to demonstrate is that even without any other circumstantial or exogenous constraints have been taken away, there is always a more fundamental incentive for production based on exchange-value to diverge from people's real desire for use-values. Our model shows us that even in perfect conditions, perverse incentives for capitalists to produce independently of the public will exist and have dire effects on the society.

Another obvious difference between our toy model and the real world is that in the real world these disproportionality crises are paired with actual market signals - unemployment, inflation, mass defaulting on debt, panic selling in the stock and bond markets, and so on. Of course, this is where our theoretical toy model falls short. However, this need not be considered a weakness of the model, since, as we've already noted numerous times, it demonstrates that these disproportionality crises are *not* representative of the imperfections of a capitalist system, but are in fact a feature of the system's proper functioning.

Thus I have come to believe (after a significant amount of doubt) that these disproportionality crises observed within this completed model for volume II do not merely serve Marx's purpose on the side of critiquing political economy, but also demonstrate important laws of motion regarding capitalism's dynamics. They demonstrate not only the social necessity of crisis, but also something about the character of such crises. They show that the basic contradictions between exchange-value and use-value, combined with the desire for accumulation at all costs guarantee periodic crises of disproportionality and make certain the irresponsibility of the capitalist class when it comes to the ability to coordinate.

In the context of post-capitalist societies, they demonstrate the dangers of over-reliance on a single metric in order to

To clarify the difference between our periodic crises of disproportionality and the crisis of overproduction, let us next ask the question: will the inevitable bursting of the AI bubble be the end of capitalism? Unfortunately, this seems very unlikely. In contrast, I propose we see the crisis of overproduction as a dark shadow cast over capitalism which periodically manifests on a much longer time-scale. The apocalyptic crisis of World Wars I and II, the profit-squeeze crisis of the late 1960s, and the 2008 financial crisis, I believe are examples of what the crisis of overproduction looks like. In the years leading up to WWI, the European imperialist blocs had just finished carving up the world. As Luxembourg noted, this imperialist project had served the purpose up to then of creating new consumer markets which the productive forces could dump their finished goods onto. When there were no new markets to create, increasing labor productivity finally threatened the viability of the system. The crisis of overproduction thus manifested as an attempt by each European power to cannibalize the others.

The profound crisis of 2008, which in many ways we are still in the throes of, I also believe is a crisis of overproduction. Following World War II, workers in the so-called first world had achieved enough political power that they could function as the primary consumer markets of capitalism. This, along with the need of rebuilding a destroyed Europe, and in industrializing the rest of the world, was enough to defer the crisis of overproduction. However, with the advent of neoliberalism, this consuming power was eroded to the point where the crisis began to reassert itself in 2008.

An adequate Marxist theory of crisis, I believe, needs to distinguish between those which pose existential threats to the system, and those which occur as the downturns of what is typically referred to as the business cycle. The distinction is important, because, despite the disruption they cause, the latter category serves an important purpose for capitalism's resilience and ability to preserve itself. Acting as fuel for capitalism's adaptability are the many smaller forms of crisis which it periodically produces throught it's many internal contradictions. We have seen in this paper the necessity of intervention in order to restart capitalism on growth paths which better approximate balanced growth. If capitalism needs this type of routine intervention, is the periodic, almost systematic, production of such situations truly a mistake of capitalism? Could the overdetermined routine of these crises instead be considered a feature? These disproportionality 'crises' seem to me to be more akin to the monstrous automatic mechanism Marx describes in volume I which maintains and governs the reserve army of labor. These mechanisms are things which capitalism needs, and which capitalism subsequently produces for itself. I think it is fitting for Marx to describe such a built-in mechanism for reproducing the capital-labor relation and the reserve army in volume I, concerned as it is with the sphere of production, and then to describe another built-in mechanism for reproducing the proper circulation and accumulation of capital in volume II, concerned as it is with the sphere of circulation. It makes volumes I and II of capital feel more properly as duals to each other, which we've argued here is what Marx originally had in mind.

One of Marx's primary criticisms of this political economists was their failure to adequately acknowledge the dual character of commodities. There are major contradictions between use-value and value which are fundamental to the capitalist system, and which bourgeois economists routinely fail to acknowledge or recognize. I believe that our model here illustrates this contradiction in a truly climactic way. It serves as an argument for why growth expressed purely in terms of value will not and cannot ever run parallel with desirable growth in terms of use-values. The political economists of Marx's time were completely concerned with growth in terms of value, failing to note that if one stopped to consider what the actual vectors of growth would be if that was their only concern. Does this growth translate to production of more means of production, or of more consumer products? Are these two distinct use-values produced in the proportions needed for a stable society? Why should they be? Morishima's model shows that the growth one gets in terms of the distinction between these use-values is completely irreconcilable with a functioning society. This argument thus serves as an extremely convincing illustration of their failures, while at the same time reinforcing one of the most important overall themes throughout the volumes of capital.

The overall theme which joins together all of our observations in this paper have been the insurmountable rift between the world of exchange-value and the real world of use-values. We have seen this rift manifesting in the form of crises, and we have seen that it is fundamentally incapable of addressing the wants, needs and desires of a population - even just the employed population - in a sufficiently unmolested manner. Perverse incentives to the contrary of the needs and wants of a population exist even in perfect conditions. The final section of this paper observed a further information-theoretic aspect of this rift. It is only natural that a rift of this sort would also actively hide information about the use-value desires of a population or even of the reproduction of society from the capitalists themselves.

This argument also I believe holds valuable lessons for us in the modern world. It continues to serve as an extremely damning argument against any argument against anyone insisting that the problems we have would be solved if only our markets were 'more free'. It also serves as an argument against so-called 'market socialists' who insist that the price signal can in any way serve to coordinate societal growth in a positive way. I believe what this model shows very generally is that the price signal, and indeed any single analog signal, is never sufficient to coordinate the diverse heterogeneous needs of a society. Such an idea flies in the face of what Stafford Beer would call the law or requisite variety[1]; there is a fundamental mismatch between the complexity of something like the price signal, a single number aggregating an immense amount of information, and the multitude of needs which a society has.

A Solving the Difference Equations

The following section is not for the mathematically feint of heart, and is only here for completeness. There is no need for you to understand this solution in order to understand the contents of this paper!

Solving the system

$$y_1(t) = c_1 y_1(t+1) + c_2 y_2(t+1) \tag{112}$$
$$y_2(t) = v_1 y_1(t+1) + v_2 y_2(t+1) + b s_1 y_1(t) + b s_2 y_2(t) \tag{113}$$

will require some familiarity with linear algebra. In particular, we will need to diagonalize a matrix. Firstly, we can rewrite this system as a matrix equation:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} y_1(t+1) \\ y_2(t+1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ bs_1 & bs_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \tag{114}$$

$$= \begin{pmatrix} c_1 & c_2 \\ \frac{bs_1c_1+v_1}{1-bs_2} & \frac{bs_1c_2+v_2}{1-bs_2} \end{pmatrix} \begin{pmatrix} y_1(t+1) \\ y_2(t+1) \end{pmatrix} \tag{115}$$

$$:= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} y_1(t+1) \\ y_2(t+1) \end{pmatrix} \tag{116}$$

To diagonalize the matrix M is to find a basis on which M acts 'nicely'. If we find that, then we can re-express the system of equations in that new basis, solve it there, and then shift back into the standard one. This nice basis is the eigenbasis of the matrix, which may or may not exist. Assuming it does for the moment let μ_1 and μ_2 denote the eigenvalues of M (in desmos they are θ_1 and θ_2). Through setting the determinant of $M - \mu I$ equal to 0, then applying the quadratic formula and simplifying the contents of the radical, we get

$$\mu_1 = \frac{1}{2}(M_{11} + M_{22} + \sqrt{(M_{11} - M_{22})^2 + 4M_{12}M_{21}}) \tag{117}$$

$$\mu_2 = \frac{1}{2}(M_{11} + M_{22} - \sqrt{(M_{11} - M_{22})^2 + 4M_{12}M_{21}}) \tag{118}$$

Since all of the M_{ij}s are positive we can see immediately that μ_1 is always positive. Furthermore since the radical is always positive, μ_1 is more than halfway between the smaller and the bigger of M_{11} and M_{22}, so it must be bigger than at least one of these. Note that there exists some $u > 0$ such that

$$\sqrt{(M_{11} - M_{22})^2 + 4M_{12}M_{21}} = |M_{11} - M_{22}| + u \tag{119}$$

since this is the distance plus some extra positive stuff inside the radical. Considering μ_2 with this in mind, we see that we are taking the halfway point between M_{11} and M_{22}, and subtracting away more than half the distance between them. Therefore μ_2 may or may not be negative, but is always smaller than both of M_{11} and M_{22}. From the same argument we can also see that μ_1 is *bigger* than M_{11} and M_{22}. When μ_1 and μ_2 are both positive, it is of course the case that $\mu_1 > \mu_2$. Suppose that μ_2 is negative and that $\mu_1 < -\mu_2$. Then

$$\frac{1}{2}(M_{11} + M_{22} + |M_{11} - M_{22}| + u < -\frac{1}{2}(M_{11} + M_{22} - |M_{11} - M_{22}| - u$$

But this would imply that $M_{11} + M_{22} < 0$, which is a contradiction since all of these entries are positive. Thus we have the following facts known about the eigenvalues:

(1) $\mu_1 > 0$.

(2) μ_1 is bigger than both M_{11} and M_{22}

(3) μ_2 is smaller than both M_{11} and M_{22}

(4) $\mu_1 > |\mu_2|$.

Next let

$$\vec{m}_1 = \begin{pmatrix} m_1^1 \\ m_2^1 \end{pmatrix} \qquad\qquad \begin{pmatrix} m_1^2 \\ m_2^2 \end{pmatrix} \tag{120}$$

denote eigenvectors associated with μ_1 and μ_2. These must satisfy the equations. (In desmos these are $m_{11}, m_{12}, m_{21}, m_{22}$.)

$$(M_{11} - \mu_1)m_1^1 + M_{12}m_2^1 = 0 \tag{121}$$

$$M_{21}m_1^1 + (M_{22} - \mu_1)m_2^1 = 0 \tag{122}$$

$$(M_{11} - \mu_2)m_1^2 + M_{12}m_2^2 = 0 \tag{123}$$

$$M_{21}m_1^2 + (M_{22} - \mu_2)m_2^2 = 0 \tag{124}$$

We can see here that $M_{11} - \mu_1$ is always negative by observation (2) earlier pertaining to the μ's, and so subtracting this term over and dividing we see that $m_1^1 = xm_1^2$ for some positive ratio x. Thus we can take the vector $\vec{m}_1$ as consisting of two positive entries. Namely let $m_1^1 = 1$, so that

$$m_2^1 = \frac{\mu_1 - M_{11}}{M_{12}} m_1^1 \tag{125}$$

Since $M_{11} - \mu_2$ is always positive, subtracting the term over and dividing gives that $m_1^2 = -xm_2^2$ for some positive x, and therefore we can take $\vec{m}_2$ as having a positive first entry and a negative second entry. In particular, again taking $m_1^2 = 1$, we have

$$m_2^2 = \frac{\mu_2 - M_{11}}{M_{12}} m_1^2 \tag{126}$$

Thus $\vec{m}_1$ points somewhere in quadrant 1 and $\vec{m}_2$ points somewhere in quadrant 2, which is enough to be sure they are linearly independent. It follows that M has an eigenbasis, and thus is always diagonalizable, as long as both μ values are positive, which itself can be easily verified as the case as long as $k_1 \neq k_2$ (which is why my system breaks when you set them equal). Let

$$P = \begin{pmatrix} m_1^1 & m_1^2 \\ m_2^1 & m_2^2 \end{pmatrix} \implies P^{-1} = \frac{1}{m_1^1 m_2^2 - m_1^2 m_2^1} \begin{pmatrix} m_2^2 & -m_1^2 \\ -m_2^1 & m_1^1 \end{pmatrix} \tag{127}$$

Here seeing P and P^{-1} as change of basis matrices, P takes vectors in the eigenbasis back to the standard basis, while P^{-1} takes vectors from the standard basis into the eigenbasis. Let $\vec{y}(t)$ denote the pair of y values in the standard basis, $\vec{z}(t)$ the vector in the eigenbasis, i.e.

$$\vec{z}(t) = P^{-1}\vec{y} \tag{128}$$

Then $M = PDP^{-1}$, where

$$D = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \tag{129}$$

so that

$$\vec{y}(t) = PDP^{-1}\vec{y}(t+1) = PD\vec{y}_{\mathcal{M}}(t+1) \tag{130}$$

$$\implies \vec{z}(t) = D\vec{z}(t+1) \tag{131}$$

$$\implies \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \mu_1 z_1(t+1) \\ \mu_1 z_2(t+1) \end{pmatrix} \tag{132}$$

Letting $\eta_1 = z_1(0)$ and $\eta_2 = z_2(0)$ represent the initial values of z_1 and z_2 (in desmos they are r_1 and r_2), i.e.

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{m_1^1 m_2^2 - m_1^2 m_2^1} \begin{pmatrix} m_2^2 & -m_1^2 \\ -m_2^1 & m_1^1 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} \tag{133}$$

Thus

$$\eta_1 = \frac{m_2^2 y_1(0) - m_1^2 y_2(0)}{m_1^1 m_2^2 - m_1^2 m_2^1} = \frac{(\mu_2 - c_1)y_1(0) - c_2 y_2(0)}{m_1^1(\mu_2 - \mu_1)} \tag{134}$$

$$\eta_2 = \frac{-m_2^1 y_1(0) + m_1^1 y_2(0)}{m_1^1 m_2^2 - m_1^2 m_2^1} = \frac{-(\mu_1 - c_1)y_1(0) + c_2 y_2(0)}{m_1^2(\mu_2 - \mu_1)} \tag{135}$$

we have then that $z_i(t+1) = \frac{1}{\mu_i} z_1(t)$ for each i, so that

$$z_1(t+1) = \frac{1}{\mu_1^t}\eta_1 \tag{136}$$

$$z_2(t+1) = \frac{1}{\mu_2^t}\eta_2 \tag{137}$$

But $\vec{y}(t+1) = P\vec{z}(t+1)$, so

$$\begin{pmatrix} y_1(t+1) \\ y_2(t+1) \end{pmatrix} = \begin{pmatrix} m_1^1 & m_1^2 \\ m_2^1 & m_2^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_1^t}\eta_1 \\ \frac{1}{\mu_2^t}\eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 m_1^1 \left(\frac{1}{\mu_1}\right)^t + \eta_2 m_1^2 \left(\frac{1}{\mu_2}\right)^t \\ \eta_1 m_2^1 \left(\frac{1}{\mu_1}\right)^t + \eta_2 m_2^2 \left(\frac{1}{\mu_2}\right)^t \end{pmatrix} \tag{138}$$

Setting $1 + g_i = \frac{1}{\mu_i}$ for both i, we finally have the solutions

$$y_1(t) = \eta_1 m_1^1 (1+g_1)^t + \eta_2 m_1^2 (1+g_2)^t \tag{139}$$

$$y_2(t) = \eta_1 m_2^1 (1+g_1)^t + \eta_2 m_2^2 (1+g_2)^t \tag{140}$$

(It seems like we should decrement and have $t-1$'s in the exponent but this doesn't produce functions which line up with $y_1(0)$ and $y_2(0)$ correctly unless you don't, and I don't care enough to consider the matter further than that.) There are some important observations to make about these solutions to make sense of them. First, since $\mu_1 > |\mu_2|$, we have that

$$1 + g_1 < |1 + g_2|$$

Moreover we make the following claim:

Lemma A.1. *$\mu_1 < 1$, always. Furthermore μ_2 is negative iff $k_2 > k_1$ (where $k_i = \frac{c_i}{v_i}$).*

Proof. If we multiply equation 121 by $1 - bs_1$, multiply equation 122 by $1 - bs_2$, add the two equations together, and foil everything out into individual terms (probably actually a bad idea, see below), we get a big mess, set equal to 0. Factor out μ_1 from every term it's present in the mess. All remaining terms end up having either an m_1^1, an m_2^1, or both. Factor these out from their terms. Inside of the expression multiplied by m_1^1, without applying any identities it should turn out that it all collapses to $c_1 + v_1$. Likewise the stuff multiplied by m_2^1 should collapse to $c_2 + v_2$. Move the μ_1 stuff over to the other side of the equation, and distribute the minus sign. What we're left with is the equation

$$\mu_1[(1 - bs_1)m_1^1 + (1 - bs_2)m_2^1] = (c_1 + v_1)m_1^1 + (c_2 + v_2)m_2^1 \tag{141}$$

$$= (1 - s_1)m_1^1 + (1 - s_2)m_2^1 \tag{142}$$

since $v_i + c_i + s_i = 1$. Now since $b < 1$, we have that $bs_1 < s_1$, so that $1 - bs_1 > 1 - s_1$. Thus the left hand side of this equation would be greater than the right hand side if $\mu_1 \geq 0$. Since equality presumably holds it must follow then that $\mu_1 < 0$.

Moving on to μ_2, multiply equation 123 by $v_1 + bs_1c_1$, equation 124 by $c_1(1 - bs_2)$ (distribute but don't foil!), and subtract the latter from the former. Two terms will be seen to cancel. Pull μ_2 out of the appropriate terms and subtract it over to the other side, then factor m_2^2 out of what's left. The expression multiplied by m_2^2 will simplify to $v_1c_2 - c_1v_2$, so that we are left with the equation

$$\mu_2[(v_1 + bs_1c_1)m_1^2 - c_1(1 - bs_2)m_2^2] = (v_1c_2 - c_1v_2)m_2^2$$

We know that m_1^2 is going to be positive while m_2^2 is negative. Therefore the whole expression which μ_2 is multiplied by is positive. The significance of this is that the sign of the left hand side is completely determined by μ_2. With this in mind, consider the right hand side. Here, again m_2^3 is negative, so this side's sign is the reverse of the sign of the expression $v_1c_2 - c_1v_2$. Thus μ_2 is positive iff $v_1c_2 - c_1v_2$ is negative, and negative iff $v_1c_2 - c_1v_2$ is positive. But since these variables are all themselves positive, we have

$$v_1c_2 > c_1v_2 \iff \frac{c_2}{v_2} > \frac{c_1}{v_1}$$

□

Since μ_1 is positive and less than 1, it follows that $1 + g_1$ is positive and greater than 1, i.e. $g_1 > 0$. If the composition of capital for the wage goods industry is greater than that of the capital goods industry, then

$\mu_2 < 0$ and so $1 + g_2 < 0$ as well. Moreover we will have $1 < 1 + g_1 < |1 + g_2|$, so we have that $1 + g_2 < -1$, i.e. the growth cannot decay over time - it must blow up, as it oscillates every period.

Next let's note that since m_1^2 is always the opposite sign as m_2^2, it is always the case that the sign of the second term in the first equation must always be different from the sign of the second term in the second equation. Thus even when $1 + g_2 > 0$, it will always be the case that $y_1(t)$ and $y_2(t)$ blow up in *opposite* directions.

From this we can make the critical conclusion about balanced growth paths: the only solutions here in which $y_1(t)$ and $y_2(t)$ both stay positive are the ones in which $1 + g_2 = 0$, which itself can only happen if $\eta_2 = 0$. This itself requires that $y_1(0)$ and $y_2(0)$ satisfy the condition:

$$y_1(0) = \frac{m_1^1}{m_2^1} y_2(0) \tag{143}$$

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