Empirical Redemption of Marx's Law of the Tendential Fall in the Rate of Profit Through Computer Simulations of Dynamic Cross-Dual Disequilibrium Models*

Alex Creiner

Abstract

In the past few years, the efforts of many theorists to develop convergent dynamic models of classical gravitation have come to fruition. These complex models, more than simply vindicating the classical theory, open the door to the scientific investigation of economic phenomena via computer simulation. In this paper, we begin that process, with the aid of an open-source sister application developed in tandem with this paper. Using this software, we investigate the effects of changing techniques of production within cross-dual models with varying output, price, and interest rate dynamics. These changes are applied both in the case of discrete 'shocks' in which the system is allowed to settle back into equilibrium, as well as in the case of continuous changes and persistent disequilibrium. In all cases considered, we empirically verify that Marx's argument for a technologically induced falling rate of profit appears to hold up remarkably well, in spite of a wealth of prior results indicating the exact opposite. In the case of interest rates which float proportionally to the relative changes in capitalist savings, we find that labor saving, capital using, cost reducing technical changes have the unwavering and unique effect of causing the rate of profit to fall, both in the continuous and the discrete cases. In the case of fixed interest rates, we find that the effect persists, but only as a pure disequilibrium phenomena in response to continuous technical change. Paired with these empirical results is a thorough investigation of why the effect occurs, and some reflection on what simplifications of the classical model allowed for these results to elude theorists for so many years.

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1 Introduction and Methodology

The classical political economists, as enlightenment thinkers, found it natural to assume that all systems eventually would come to 'rest' barring any exogenous intervention. From this thinking arose a notion of equilibrium relating to capitalism. The key features of this classical equilibrium were perpetual supply and demand equilibrium, and, critically, equal profit rates across all sectors of the economy. To them, the latter condition is a necessary condition for the former, for if profit rates were to differ between two sectors, then capitalists would elect to shift their capital into the more profitable sector, leading to supply-demand mismatches and a disruption of equilibrium. [1][16] To express this in a more modern way, deviations of profitability above the average drive up ouput, in turn driving down prices and dragging profits back down, while profit rates below the average drive down output, driving up prices and pulling profits back up. Thus capitalism, as a disequilibrium process approaching equilibrium was one in which the *cross-dual* interactions between price and output creates a 'graviational pull', gradually bringing all sectoral profit rates towards one another and meeting at a uniform average.

This classical notion of equilibrium has the advantage of being dynamical in nature - based on assumptions about how a system would evolve over time based on the actual ways in which prices, producers and consumers can be naturally expected to behave. This stands in stark contrast to the much more popular Walrasian neoclassical equilibrium, which is maintained essentially by theoretical fiat (i.e. the assertion of market pauses and auction systems adjusting prices until equilibrium is reached [20]). However, this advantage is also likely why it was largely abandoned through the first half of the 20th century. The story of classical equilibrium, while simple, amounts to a very non-trivial claim about complex dynamical systems which require a great deal of mathematical sophistication to handle rigorously. Efforts to demonstrate the convergence towards classical equilibrium within formal mathematical models did not begin until around the 1980s.[8] Since then, a number of cross-dual models have been proposed to investigate the convergence towards classical equilibrium by the likes of Nikaido, Duménil and Lévy, Kuroki, Boggio and many others.[14][10][2][4] While the initial results of these theorists were rather pessimistic towards the classical gravitation theory,[13] more recent work by Kiedrowski[8], Bellino and Serrano[1], and especially Wright[21][22] have thoroughly redeemed the concept, and shown that the pull towards a classical equilibrium with uniform profit rates is a very real process which occurs under a variety of very general assumptions and in a variety of different models.

Wright's model is especially noteworthy, because while most of the proposed cross-dual models are singularly focused on the dynamics of price and output, his chooses to incorporate many other features of a capitalist economy such wages, consumption habits, class relations and credit dynamics, all of which are constrained by a finite supply of labor and money. Wright is currently actively developing the model further, and has since modified it to allow for joint production, fixed capital and heterogeneous labor. While it has not yet been formally proven converge of his system towards equilibrium for any particular class of technical parameters and initial conditions, he (and since, I) has confirmed through extensive computer simulation testing that the model appears to be stable and demonstrate equilibrium convergence in nearly all cases.

Having these complex dynamic models, however, is only the first step. What remains is to *use* them in order to test and investigate all manner of hypotheses about capitalist economies. In this paper, we begin that process. In a companion application to this paper, I have converted his model into a steppable and *perturbable* three-commodity capitalist system 'in a box' subject to experimentation[2] Users can apply all

[1]Smith: *If, in the same neighbourhood, there was any employment evidently either more or less advantageous than the rest, so many people would crowd into it in the one case, and so many would desert it in the other, that its advantages would soon return to the level of other employments.*[19]

[2]Wright's model allows for an arbitrary number of commodity types, but three is a good manageable number to investigate complex behavior. The code was written with the intent of making it easy to modify the application to accomodate higher

manner of shocks to the economy and observe visually how all relevant quantities respond at once. The app is open source and can be found on GitHub here, with binary releases available for non-programmers. The GUI interfaces is designed to be easily repurposable for investigating models of all kinds, written in any language. I strongly encourage readers to download the application and have it open alongside their reading of this paper, and will proceed under the assumption that this is the case. The primary results that I have to share pertain to how the rate of profit changes in response to technological changes of different kinds. I will show that Okishio's theorem is all but irrelevant in it's implications for the behavior of these cross-dual models in response to these changes. Indeed, our empirical results will offer a near-total redemption to Marx's argument for a technologically induced falling rate of profit; the introduction and generalization of capital-using, labor-saving cost reducing technological changes to an economy will indeed have the unique net long term effect of reducing the equilibrium and sectoral rates of profit, contrary to what the classical fixed real-wage models have indicated previously.

My implementation of Wright's model, like his own computer simulations, uses the backwards differentiation algorithm. This is an iterative technique which repeatedly takes the slope between it's previous two steps as an approximation for the derivative, and then follows that slope, approximating the function with it's tangent line. Where my simulation system differs from his is in its use of time steps. After simulating what amounts to a single period, it stops itself, allowing for interventions in the form of either discontinuous shocks or conditional changes to the differential equations themselves. This effectively turns the model into a laboratory fit for experimentation of all kinds.

2 Overview of the Model

2.1 Definition of the Model

We must begin with a brief summary of the model and existing results pertaining to it.[3]. The model assumes away fixed capital and joint production. All capital is circulating, and all commodity types have production times which equal their working times (i.e. there is no waiting for crops to grow, etcetera). Labor is homogeneous. There is no differentiation among professions, all workers are equally productive, and receive the same hourly wage w.[4] Wright's model also assumes a fixed money supply M, a fixed amount of available labor L (measured in hours), and fixed technology (i.e. fixed technological and living labor coefficients a_{ij} and l_i). We will remove these assumptions and allow these quantities to be altered, not by any assumptions about their natural dynamics, but rather via exogenous 'shocks'.

The linear algebra notation I employ will be that which is standard to the mathematics or the physical sciences: all vectors are assumed to be column vectors unless otherwise noted, with matrix multiplication happening from the left. Our economy has n distinct commodity types. Let $A \in \mathbb{R}^{n\times n}$ denote the Leontief input/output requirements matrix, $\mathbf{l} \in \mathbb{R}^n$ the living labor vector, and $\mathbf{p} \in \mathbb{R}^n$ the unit price vector. At the start of a given period t, we also have a total output bundle $\mathbf{q}$[5] and a supply bundle $\mathbf{s}$. Workers as a whole have at a given moment a total money savings amounting to m_w, and capitalists have their own total savings m_c. We assume the existence of constants $0 \leq \alpha_w, \alpha_c \leq 1$ representing these classes' propensity to consume. At the start of a period (or over the course of the previous period, depending on how you want to conceptualize it), workers and capitalists purchase goods out of the available inventory $\mathbf{s}$. Workers will spend the proportion of their savings m_w given by α_w, and capitalists will do the same according to their savings m_c and the proportion α_c. We assume the existence of the fixed 'proportional bundles' $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ defining these two classes' consumption habits. As mentioned, workers and capitalists will spend money amounting to $\alpha_w m_w$ and $\alpha_c m_c$ respectively, and they will spend it purchasing exactly as much of a scalar multiple of the vectors $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ as they can purchase with that money. Thus if $\mathbf{b}$ and $\mathbf{c}$ denote the total bundle of goods consumed by workers and capitalists respectively, then

numbers if one wishes to do so.

[3]for a more detailed presentation one should see either Wright (2015)[22] or Wright (2017)[21]

[4]As noted already, Wright himself has steadily modified his system to remove many of these simplifying assumptions. Most notably his system now allows for fixed capital, joint production, and heterogeneous labor.[7][6]

[5]Since this 'output bundle' is a continuously evolving quantity, it would be more appropriate to refer to it as a vector of activity levels. Referring to it as output is only for the sake of conversation.

$$\mathbf{b} = \frac{\alpha_w m_w}{\mathbf{p} \cdot \underline{\mathbf{b}}} \underline{\mathbf{b}} \qquad\qquad \mathbf{c} = \frac{\alpha_c m_c}{\mathbf{p} \cdot \underline{\mathbf{c}}} \underline{\mathbf{c}} \tag{1}$$

m_w, m_c and $\mathbf{p}$ are all dynamically evolving functions of time - we are simply omitting the function notation for notational simplicity. On the other hand, $\underline{\mathbf{b}}, \underline{\mathbf{c}}, \alpha_w$ and α_c are all assumed constant. Total demand (i.e. both productive and unproductive consumption) over the course of a period is the sum of the two consumption bundles along with the means of production bundle $A\mathbf{q}$:

$$\mathbf{d} = A\mathbf{q} + \mathbf{b} + \mathbf{c} \tag{2}$$

This bundle is continually being deducted from the output vector $\mathbf{q}$ (i.e. newly produced goods are consumed 'first') while the remainder of $\mathbf{q}$ is added to the overall supply $\mathbf{s}$, thus giving us our first set of differential equations

$$\frac{ds_i}{dt} = q_i - d_i \tag{3}$$

where s_i, q_i and d_i denotes the i^{th} entry of $\mathbf{s}, \mathbf{q}$ and $\mathbf{d}$ respectively, $i = 1, \ldots, n$.

Turning next to price dynamics, the relative price of any commodity is assumed to be inversely proportional with the relative supply of that commodity:

$$\frac{1}{p_i} \frac{dp_i}{dt} = -\eta_i \frac{1}{s_i} \frac{ds_i}{dt} \tag{4}$$

Where $\eta_i > 0$ is a constant of elasticity. Thus as the relative supply of a commodity goes up, relative prices fall, and as supply goes to zero, prices rise toward infinity. Wright's choice of a persistent inventory of excess supply is one of the core features of his model which distinguishes it from other cross dual models, which tend to adjust prices in direct response to the output. The more typical cross-dual formulation for price adjustment is

$$\frac{1}{p_i} \frac{dp_i}{dt} = -\eta_i \left(\frac{d_i - q_i}{q_i} \right) \tag{5}$$

Here, prices fall the moment that output exceeds demand. The introduction of a supply 'buffer' allows for prior production to intervene and cover for supply-demand mismatches, allowing for the system to adjust more smoothly.

Wages are treated no differently from any other commodity. Recalling that the total available labor at any moment in time is L, and noting that employment is $E = \mathbf{l} \cdot \mathbf{q}$, rule 4 applied to the special case of wages amounts to the assertion

$$\frac{1}{w} \frac{dw}{dt} = \eta_w \frac{1}{L - \mathbf{l} \cdot \mathbf{q}} \frac{d(\mathbf{l} \cdot \mathbf{q})}{dt} \tag{6}$$

where $\eta_w > 0$ is a constant of elasticity. Note here that $L - \mathbf{l} \cdot \mathbf{q}$ essentially represents the size of the 'industrial reserve army'.[6] Turning to the capitalists, a key simplifying assumption of the model is that we choose not to have the capitalists paying out of pocket for the capital used in production. Rather, capitalists *finance* the entirety of their productive activities by borrowing at a floating singular interest rate $r(t)$. Each period, they are expected to pay back a proportion of their costs given by this rate, so that each sector has a total effective cost of production (i.e. the total money capital employed in production)

[6] In Wright's original model, the quantities $\mathbf{l}$ and L are both assumed constant with time. In this special case, w can be seen as a function of $\mathbf{q}$ alone and this differential equation can be solved in closed form. However, we are making no such assumptions for either variable.

$$(1+r)(\mathbf{a}_i \cdot \mathbf{p} + wl_i)q_i \tag{7}$$

where $\mathbf{a}_i$ denotes the i^{th} column of A. There are no requirements that the total money capital does not exceed the total actual money supply M, and in practice it nearly always does. Thus conceptually we have a developed financial sector which is capable of accomodating any level of commerce. Each period then, capitalists (in particular finance capitalists) receive a total interest income in the amount $\Psi = r(A\mathbf{p}+w\mathbf{l})\cdot\mathbf{q}$. In addition to interest, capitalists of course receive income in the form of profit of enterprise, which is simply the difference between total revenue and total cost. The total profit of enterprise for the i^{th} sector is therefore

$$\Pi_i = p_i d_i - (1+r)(\mathbf{a}_i \cdot \mathbf{p} + wl_i)q_i \tag{8}$$

The sum of all of these Π_i plus the total interest income Ψ amounts to the total income of the capitalist class as a whole. This income is continually added to the capitalist savings while consumption is continually subtracted. We therefore have the differential equation representing the change in capitalist savings over time:

$$\frac{dm_c}{dt} = \Psi + \sum_{i=1}^{n} \Pi_i - \alpha_c m_c \tag{9}$$

As noted, the interest rate $r(t)$ is itself a floating quantity in Wright's model, and rises or falls according to the supply and demand of money capital. Here the supply of money is equivalent to the total money possessed by the capitalists, m_c. We therefore assert:

$$\frac{1}{r}\frac{dr}{dt} = -\eta_r \frac{1}{m_c}\frac{dm_c}{dt} \tag{10}$$

where once again $\eta_r > 0$ is a constant of elasticity. Thus, while money is not borrowed directly from the capitalist savings, the 'price' of money capital nonetheless rises or falls according to it. Wright's decision to model interests rates this way is in adherence to Marx's own theory, but a more modern understanding of interest rates would acknowledge that since banks tend to possess the *entire* money stock at any given moment, the availability of money capital is independent of the distribution of money between the two classes. Thus a more realistic model of the interest rate over time would simply be

$$\frac{dr}{dt} = 0 \tag{11}$$

Luckily, this is just the special case of 10 in which $\eta_r = 0$. We will primarily direct our focus to this case.

Finally we turn to the adjustment of output. Our model asserts that outputs adjust according to profitability. As long as a sector is profitable, capitalists will continually increase the the relative scale of the production for it, and conversely when a sector is unprofitable, they will pull out, and the relative scale of production will fall. The original model asserts

$$\frac{1}{q_i}\frac{dq_i}{d_t} = \kappa_i \pi_i \tag{12}$$

Where $\kappa_i > 0$ are once more constants of elasticity, and π_i is the *rate* of profit for sector i, (i.e. Π_i divided by the total unit cost including interest). In more recent iterations of the model Wright has advocated instead using the equation

$$\frac{dq_i}{dt} = \kappa_i \Pi_i \tag{13}$$

In general the behavior of the model is qualitatively the same for both equations, but equation 12 tends to be the more volatile one. We will heed Wright's advice and default to equation 13, but the simulation software allows users to choose between the two using a dropdown near the top of the controlsm, and there will be one point in the paper where the first demonstrates qualitatively different behavior that will deserve some commentary.

Equations 4 and either 12 or 13 assert a cross-dual form in which price imbalances lead to changes in output while imbalances in output lead to changes in prices. However, his formulation of the output equations again differs from the standard cross-dual models, which tend to assert either

$$\frac{1}{q_i}\frac{dq_i}{dt} = \pi^* + \kappa_i(\pi_i - \pi^*) \tag{14}$$

or

$$\frac{1}{q_i}\frac{dq_i}{dt} = \kappa_i(\pi_i - \pi^*) \tag{15}$$

where

$$\pi^* = \frac{\Pi}{(1+r)((A^T\mathbf{p} + w\mathbf{l}) \cdot \mathbf{q})} \tag{16}$$

is the overall *average* rate of profit. The weakness of these equations, as pointed out by others such as Kiedrowski[8], is the assumption that a capitalist has existing knowledge of what this average is for the sake of comparison. This is an inherited weakness of the classical story as a whole, and a precursor to many of the 'perfect knowledge' assumptions which are often made in neoclassical economics. In Wright's formulation, no special knowledge is required of the capitalist aside from their own business outcomes. Output increases as long as a sector is profitable, regardless of the state of the other industries.

2.2 A Running Example

Figure 1 depicts an example of the basic dynamics of the system in the case of a three-commodity economy. Wright chooses to call his three commodities corn, iron, and sugar, and we follow his lead. The parameters defining this instance of the model are exactly the same as those which he chose for his thesis.[7] Corn requires equal small amounts of corn and iron, and a fair bit of labor. Iron requires quite a bit of iron, a tiny bit of sugar, and a bit less labor than corn. Sugar requires some corn and a bit of sugar,but no iron and significantly less labor than corn and iron. The society starts out with a shortage of iron (note the initial price spike). Workers consume mostly corn and some sugar, while capitalists consume mostly sugar and some corn. The reader is encouraged to tweak the numbers for themselves to get a sense of how the model behaves more generally.

We will not exhaustively list every number here - the exact numbers are easily visible within the application, or can be found in the 'params.py' file found in the GitHub repo. Additionally, a glossary of all equations used for plots can be found here.

2.3 Prior Results and Initial Observations

We now highlight some preliminary observations and results pertaining to the model which will be important for our analysis going forward.

[7] At some point, I must have accidentally set the sugar requirements for iron to 0.1, where in his example it is set to zero. This is as far as I can tell the only difference.

2.3.1 Equilibrium Convergence

As we can see, the system tends to settle into supply-demand equilibrium for most if not all 'viable' input parameters.[8] I have included in the application a function which generates random parameters, to confirm this in the case of a three-commodity economy. The reader is encouraged to roll a few 'random economies' and see this behavior for themself.

For simplicity throughout this paper we let

$$\mathbf{m} = A^T\mathbf{p} + w\mathbf{l} = M^T\mathbf{p} \tag{17}$$

denote the vector of unit money costs for each commodity type. Sectoral unit profits here are defined as the difference between the price vector at time t and the cost of production at time t *excluding* cost of interest, i.e. $\pi = \mathbf{p} - \mathbf{m}$. These can be seen as profit rates from the perspective of the capitalist class as a whole, irrespective of whether that profit is financial or enterprise in nature.[9] The equilibrium rate of profit is given by finding the Perron-Frobenius maximal eigenvalue of the matrix $M = A + \mathbf{b}\mathbf{l}^T$ where $\mathbf{b}$ is the hourly real wage $\mathbf{b} = \frac{w}{\mathbf{p}\cdot\underline{\mathbf{b}}}\underline{\mathbf{b}}$, then inverting it and subtracting 1.

This equilibrium also always coincides with an approach towards equal profit rates for all sectors, thus vindicating the classical gravitation theory. The claim that the system converges to classical equilibrium for all 'reasonable' inputs is unproven, but seems to hold up perfectly to empirical testing. The reader can also choose to switch the simulation to instead use the cross-dual equations 5 and 14. The model appears to be convergent under these alterations, which is a bit surprising because they were largely thought to be fundamentally unstable. This remains the case even with interest rates set to zero (effectively eliminating the credit system from the model) and with $\eta_w = 0$, fixing the cost of labor. It is unclear what other elements of the model are forcing the system to converge in spite of these prior results. Despite this, they still have some shortcomings. In particular, there appears to be nothing stopping outputs from going negative under equations 5 and 14. I won't dwell much more on these revisions, and will simply say instead that all empirical results that I have to share appear to be independent of the particular price or output equation used. The user can easily show this to themselves within the application by simply editing the Equation Modifiers dropdown.

2.3.2 Zero Profit of Enterprise

Observe in figure 1 that once in equilibrium, total profit of enterprises invariably tends toward zero, leaving the capitalist class entirely dependent on the interest rate for their continued existence. The reason for this is plain to see by inspection of either 12 or 13. Equilibrium implies that outputs are no longer changing with time, and this can only occur when profit of enterprise is uniformally zero for all sectors. Since the change in interest is assumed to be relative, a starting interest rate of 0 will cause the interest rate to remain at 0, effectively deleting the credit system from the model. The model's interpretation then shifts to 'communism among the capitalists' in which capital is freely drawn from a communal resource pool. In this case, the capitalist class eventually disappears altogether (i.e. their savings and consumption both go to zero), and the economy devolves into Adam Smith's 'early and rude state' of 'simple commodity production'. As Smith himself would have predicted, prices and values coincide here.[10]

Thus it is a general result of these cross-dual models that profit of enterprise is purely a disequilibrium phenomenon. However it must also be noted that this model also assumes a fixed labor force, unchanging technology, and zero extrinsic sources of demand. Thus economy growth and capital accumulation are currently absent from the model. It remains to be seen if the industrial capitalists can preserve their sovereignty and role within the system in the presence of these features. This will be the subject of a future paper.

[8] By viable, we mean that the initial conditions are such that the initial labor pool is big enough to satisfy the employment level implied by output vector and living labor vector and that the initial supply vector is big enough to accomodate the demand vector.

[9] Actual enterprise profit rates will be clearly denoted as such when utilized, and given dotted lines to distinguish them from these. Enterprise profit rates are of corurse defined as the ratio of equations 8 and 7.

[10] To be specific, we have here that $\mathbf{p} = w\lambda$, where λ is the unit value vector. Plots of values are automatically normalized to be comparable with prices in the Prices and Values category of the application via multiplication by the wage w.

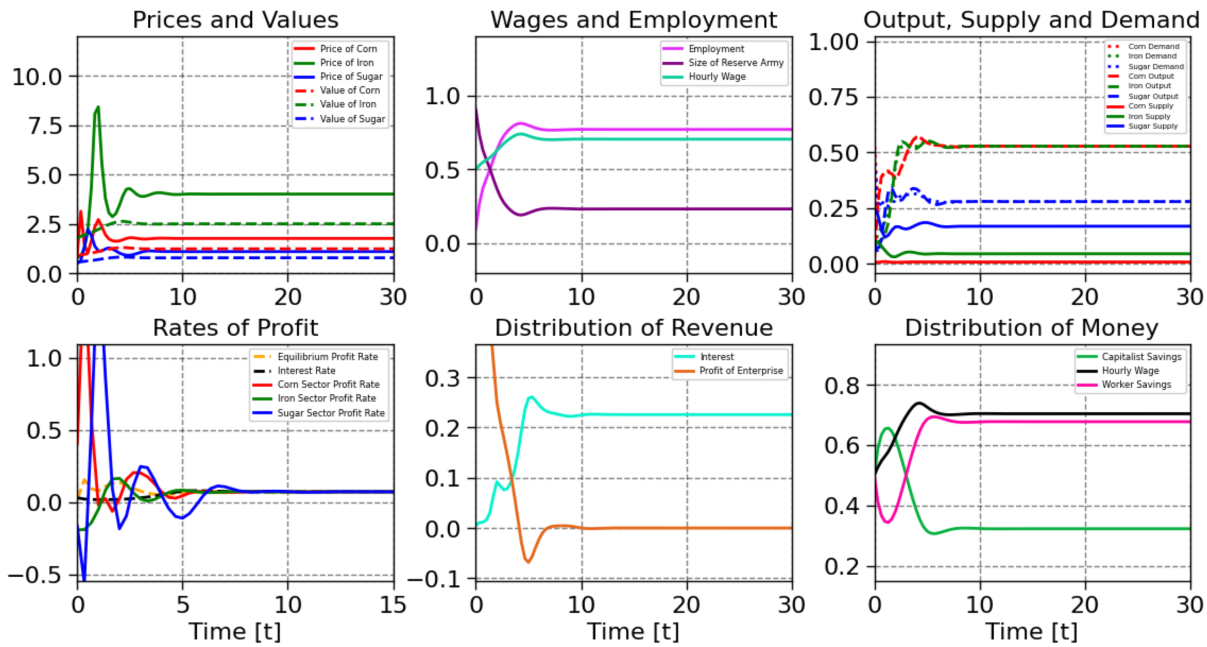


Figure 1: Basic dynamics for a 3-sector economy.

2.3.3 Zero-Sum Distribution of Money

At any moment, all money must distributed in the savings of either the workers or the capitalists, i.e.

$$m_w + m_c = M \tag{18}$$

See Wright (2015)[22] chapter 7 lemma 1 for a proof. Thus the money distribution is zero-sum between the workers and capitalists. What capitalists lose, workers must gain, and vice versa.

2.3.4 Equilibrium Incomes

Suppose that the system is settled into equilibrium, i.e. $\frac{dp_i}{t} = 0$ and $\frac{dq_i}{dt} = 0$ for all $i = 1, \dots, n$. Then workers spend what they earn:

$$\alpha_w m_w = w(\mathbf{l} \cdot \mathbf{q}) \tag{19}$$

while capitalists ‘earn what they spend’:

$$\alpha_c m_c = r(\mathbf{m} \cdot \mathbf{q}) \tag{20}$$

See Wright (2015)[22] chapter 7 lemmas 5 and 6 for proofs.

3 Okishio’s Theorem and the Classical Model

Capitalists, pressured by the coercive laws of competition, seek to obtain extra profit by reducing their costs through the research and development of new scientific techniques of production. A variety of soft-constraints guides and biases the directionality of these changes. The first of these is of course that all innovations must be *cost-reducing* for the capitalists. Second, when a capitalist seeks cost-reducing technologies, they must choose which costs to try and reduce. Given the choice between reducing the iron needed in production and reducing labor, the capitalist should be expected to typically choose the latter, because iron cannot go on strike. As long as a labor-saving innovation is cost-reducing, these changes in technique may require an overall increase in inputs of other kinds. Thus in general the most commonly expected technological changes within a capitalist economy are *capital-using and labor saving* (CULS). Of course, there are three other categories of cost-reducing technological change: *capital-saving and labor-using* (CSLU), *labor-saving* (LS), and *capital-saving* (CS).[11]

Marx claimed in volume III of Capital that the development of technology under the course of capitalism's history would lead to an overall fall in the equilibrium rate of profit.[11] This was widely taken for granted until, in 1962, Okishio showed that with the real wage fixed, *any* cost-reducing technological changes to the economy would be guaranteed to lead to an *increase* in the equilibrium rate of profit.[15] Since then, the theory of a technologically induced rate of profit has been widely discredited in academic circles, despite the efforts of many Marxists to rehabilitate it over the years.

A closer look at the theorem and the implicit model it is embedded in is in order. Let A and $\mathbf{l}$ denote the input/output matrix and living labor vector for a particular economy. Assume that all workers receive the same hourly wage w and all purchase the same real-wage bundle of commodities $\mathbf{b}$ using this money. If we assume a uniform rate of profit, then the unit price vector $\mathbf{p}$ and the uniform rate of profit π must satisfy the equation

$$\mathbf{p} = (1+\pi)(A^T + \mathbf{l}\mathbf{b}^T)\mathbf{p} \tag{21}$$

where $\mathbf{p}$ is a strictly positive right-eigenvector of the non-negative matrix $M = A + \mathbf{b}\mathbf{l}^T$ with eigenvalue $r = \frac{1}{1+\pi}$. Perron-Frobenius theory dictates that any positive eigenvector of a non-negative matrix must be associated with specifically the maximal non-negative eigenvalue of that matrix. Thus there is a single possible uniform profit rate for such a system, and it is inversely related to the maximal eigenvalue of the augmented requirements matrix M. It can be shown that, when any cost-reducing technical changes are introduced, assuming that $\mathbf{b}$ remains the same, this corresponds to a smaller maximal eigenvalue of the new matrix M', and therefore a larger uniform rate of profit.[12]

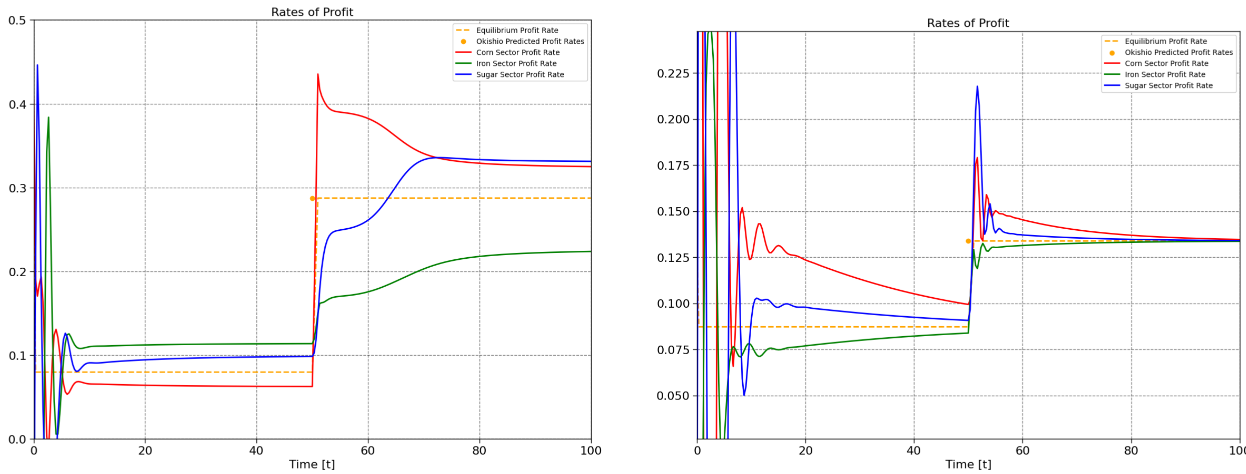
With some blunt-force alterations, it is not hard to witness the implications of this result within our model. To do this we re-interpret $\underline{\mathbf{b}}$ as the real wage and substitute $\mathbf{p} \cdot \underline{\mathbf{b}}$ everywhere that the hourly wage is used. We also set the initial interest $r(0) = 0$ since the classical models do not assume a credit system. This can be done in the application by selecting 'fixed real wage' under the Model Restrictions dropdown. The parameters for reproducing all figures in this paper are easily loadable within the app by selecting Parameters -> Parameter Presets -> The figure number and selecting Load parameters. There are two versions of the demonstration, one for each of the output equations 12 and 13. Figure 2 depicts the effects of a 50% reduction in living labor requirements being applied to the corn sector at time $T = 50$. The sector receiving the changes can be selected, or alternatively one can have the sector chosen at random. For demos utilizing random selection, new simulations can be executed by pressing F5.

As expecteed, Okishio perfectly predicts the trajectory of the rate of profit in both cases. In the case of the model utilizing the more recent output equation 13, we see that profit rates are no longer able to equalize. This is to be expected, since severing the signal of a changing real wage from the systems which adjusts outputs and prices profit rates serves to greatly weaken any 'gravitational pull' which might have existed. Output equation 12 is more volatile but also more sensitive, and manages to exhibit converge despite in spite of this.

Regarding 2a, it is important to witness that *despite* the lack of *actual* convergence to the equilibrium rate of profit, it is impossible not to recognize the steering effect that it has on the actual sectoral. The sectoral profit rates do not converge to the equilibrium profit rate, but they *nonetheless move in tandom with it* - all sectoral rates rise to stay within its orbit, regardless of whether they converge or not. Some theorists such as Cockshott and Cottrell[3], as well as Farjoun and Machover[5], have criticized the equilibrium theory entirely claiming with the argument that since profit rates do not actually equalize in reality, this whole business of equilibrium theory is irrelevant. These observations make it clear that these classical equilibrium

[11] There is actually a secret and forbidden fifth kind: capital saving *and* labor saving: (CSLS). This type of technical change will only be discussed in footnotes, so as not to attract too much attention.

[12] For a detailed presentation of these mathematical facts, see Roemer [17].



(a) Using output equation 13

(b) Using output equation 12

Figure 2

theories have quite a lot to say about the economic system, even one in a state of perpetual disequilibrium. Thus the lack of convergence towards equilibrium in real life does not on its own invalidate or minimize the observed results the equilibrium theory.

Other theorists have criticized the Okishio by insisting on a difference between a static equilibrium and a dynamic equilibrium. For example, Andrew Kliman in *Reclaiming Capital*[9] insists that Roemer (who undertakes a very thorough examination of the theorem and its implications in *Analytical Foundations of Marxian Economic Theory*[17]) fails to demonstrate that when a dynamic equilibrium is reached, that it will necessarily equal the static equilibrium. This critique is misguided. Roemer is emphatic throughout the book that he regards the classical equilibrium as purely static. The reason that Roemer's language slips occasionally into a dynamic framing during his presentation of Okishio is because, under the particular assumptions of the theorem, the *only* equilibrium which *can* be reached under the assumptions that have been made *is* the static equilibrium, as both figures 2a and 2b make clear. There is simply no room within the premises of Okishio's theorem for any distinction between a static and dynamic equilibrium. To the extent that a dynamic equilibrium exists at all in the classical model, they are the same.

Despite Kliman's specific argument against Okishio being misguided, his broader criticisms of the conclusions which many economists took away from the theorem are valid. Within a fully complex dynamical system where no quantities in which all variables are free to float and interact with one another in natural ways, there absolutely *could* be counterveiling emergences which render the results of Okishio's theorem irrelevant to the overall behavior of the system. To get a sense of what these shortcomings might be, let us take a closer look at the classical dynamical model we have just created. This analysis will choose to focus on the version of the model which uses output equation 12, since the dynamics here are more pronounced.

Firstly, let us ask: why do profit rates not tend to zero with zero interest rates, like they do in the unaltered model? The key is the real wage vector $\underline{\mathbf{b}}$, whose entries are initially set low enough that the price of labor ensures profitability. As long as this is the case, output and employment will begin to increase. In figure 3 we see that output consistently exceeds demand. As a consequence, prices are continually falling toward zero. However, since the real wage is fixed, this means that the money wage is also consistently falling towards zero as well. This ensures continued profitability across the sectors even as prices drop. Thus we are faced with the rather fascinating phenomenon of convergence towards a uniform profit rate and equilibrium prices despite perpetual supply-demand disequilibrium and perpetually changing prices. Prices fall towards zero, but as this happens they simultaneously approach proportions with one another which bring it towards the ray in $\mathbb{R}^n$ spanned by the Perron-Frobenius equilibrium price vector.

More to the point, the real problem with the fixed real wage assumption is that it removes from the model any consideration of the labor market. Inspecting the Wages and Employment plots of figure 3 shows employment rising continually, irrespective of the actual size of the laboring population. The unemployed

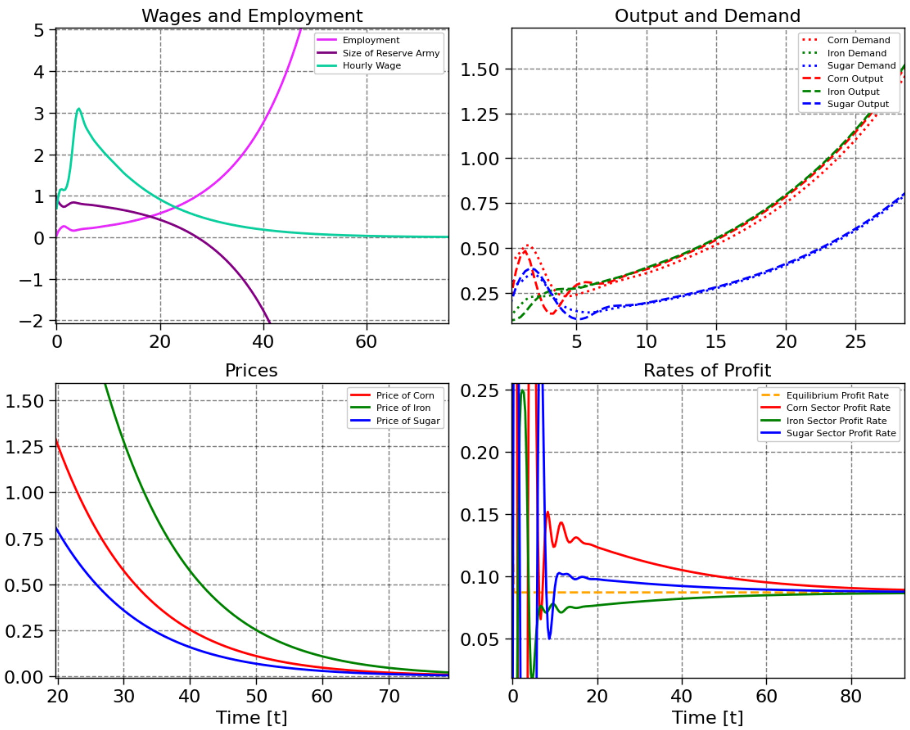


Figure 3: The fixed real wage model with outputs according to equation 12 provides a case of equilibrium prices and profit rates without convergence towards actual supply-demand equilibrium or a static price system.

population goes negative and approaches negative infinity, because without skyrocketing wages as a result of low supply, there is nothing in the model preventing this.

A few other consipicuously missing features from the classical model are the lack of any explicit dynamics regarding capitalist consumption, the money supply or interest rates. The latter is especially important in the era of cross-dual models which have revealed non-zero profit of enterprise to be strictly a disequilibrium phenomenon. The assumptions of a positive equilibrium profit rate is incompatible with the assumptions of a persistent capitalist class and zero-interest rates (which is effectively what a lack of consideration of a credit system amounts to in the absence of micro-dynamical agent-based modelling of capital investment).

The fixed real wage assumption stands out as the major theoretical shortcoming which prevents the Okishio results from truly saying anything definitive about how a real capitalist system would respond to technical changes. I have seen theorists such as Roemer and Morishima both argue that the fixed real wage assumption is valid on the grounds that it is the correct way to formalize Marx, who according to them was himself a ‘subsistence wage theorist’ observing a Victorian era British capitalism which truly did appear to hold wages at a level which is minimally sufficient for the short term biological survival of a working class.[17][12] While this may be true, does a great intellectual disservice to Marx, whose entire focus in volume I of Capital was on understanding how the process of capital accumulation simultaneously provided systemic causal mechanism which *held* the real wage fixed, *despite* it clearly varying with the size of the industrial reserve army. To simply declare the real wage fixed is to trivialize that pursuit, which to me appears antithetical to Marx’s project. Moreover, what is faithful always scientific. I am interested in understanding how the capitalist system really works, not in creating the maximally faithful formal model in which to represent Marx’s arguments.

The line of criticism of Okishio which focuses on the fixed-real wage has also been the most fruitful. Roemer himself demonstrates the rate of profit falling after CULS technical changes on the assumption that the workers in a particular sector simply maintaining wages which are proportionally the same relative to profit rates as they were before.[17]. However, many Marxists, including Roemer, have found this line of criticism insufficient for redeeming Marx’s theory, because it renders the law no longer an objective systemic outcome, but rather up to the outcome of the class struggle. However, this analysis overlooks that the outcomes of this ensuing class struggle are *not* purely subjective. While they certainly have a subjective element, they are also severely constrained by the material context of the system where they are taking place. To illustrate this within the context of our model, Wright proves that the equilibrium level of worker savings m_w^* is completely determined by the techniques of production (A and $\mathbf{l}$) and the composition of demand $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$, the propensities to consume α_w and α_c, the initial distribution of savings $m_w(0)$, and the interest rate elasticity constant η_r. In particular, they are independent market prices, the scale of production and the dynamics of the labor market. These observations will apply to the new equilibrium which is settled into after shocking the model with a technical change. If, for sake of argument, we do find that the workers are able to wrestle back sectoral shares to the point where the rate of profit falls, this will have happened passively through the natural adjustment of the various quantities within the system. In summary, it may be the case that class struggle is not, as Roemer claims, the ingredient missing from these earlier theories, but rather simply more concretely defined dynamics for how wages should change, interest rates should adjust, how money will be redistributed and so forth.

It would seem to myself as a relative outsider looking back on this historical debate as the fixed real wage assumption was made not because it had any particular basis in Marxist thinking, but rather because it allowed for fruitful mathematical inquiry which sidestepped the analytical intractibility of more complex dynamical systems which would be required to see it as a floating variable. These theorists did the best they could with the analytical tools that were available, but simplifications were necessary for the sake of abstract analysis. In the age of computer simulations, this is no longer the case, and it is time to bring Marxist economic theory into the present day. Complex dynamic models such as Wright's gives us the tools we need to get started on this.

4 Technical Changes with Floating Interest Rates

We will begin out investigation with interest rates floating according to 10. This setting is appropriate for evaluating the dynamics of the rate of profit on 'Marx's own terms', since he saw interest rates adjusting according to the distribution of income. After this, we will consider how the model changes when modelling the interest rate according to 11, which amounts to testing the theory in 'the real world'. The case of zero interest rates will be included in that investigation as a special case. Our findings in the floating case will also help us to make sense of the effects we see in the fixed case, which are more complicated.

4.1 Empirical Findings

4.1.1 Labor Saving Technical Change

Figure 4 depicts the same shock of fifty percent labor reduction (in the iron sector) in the full model where all variables are allowed to float. The story goes as follows. First, the iron sector, seeking to produce the same output as it has been but only requiring half the labor, lays off half it's workers and enjoys a sizeable spike in profit, spurring a surge of investment into the iron sector. As output increases, so does supply, causing the price of iron to quickly fall along with wages. The fall in the price of both iron and hourly wages induces a delayed spike in the profitability of the corn sector, which relies heavily on both. Meanwhile, the mass layoffs combined with the spiked profits lead to the capitalist class becoming flush with cash to spend on consumption, leading to a spike in demand for sugar (which they consume a disproportionately amount of), cutting into the supply and inducing a spike in it's own price, leading finally to a spike in profits for the sugar sector. With positive profits of enterprise now prevailing across the whole economy, the scale of production begins ramping up, leading to steadily increasing employment. As employment rises, so do wages, demand and worker savings, bringing profit rates back down across the board as the economy settles back down into equilibrium.

When the dust has settled, the overall uniform rate of profit is slightly higher than it was before, but nowhere near what Okishio's theorem predicted it would be. Okishio fails to even predict the peak of the profit spike.[13] Where the final equilibrium profit rate lands is quite mysterious, and pure labor saving

[13] The hourly real wage at the moment of the shock which is used to compute the Perron-Frobenius rate of profit is given by the equation $\mathbf{b} = \frac{w}{\mathbf{p}\cdot\underline{\mathbf{b}}}\underline{\mathbf{b}}$.

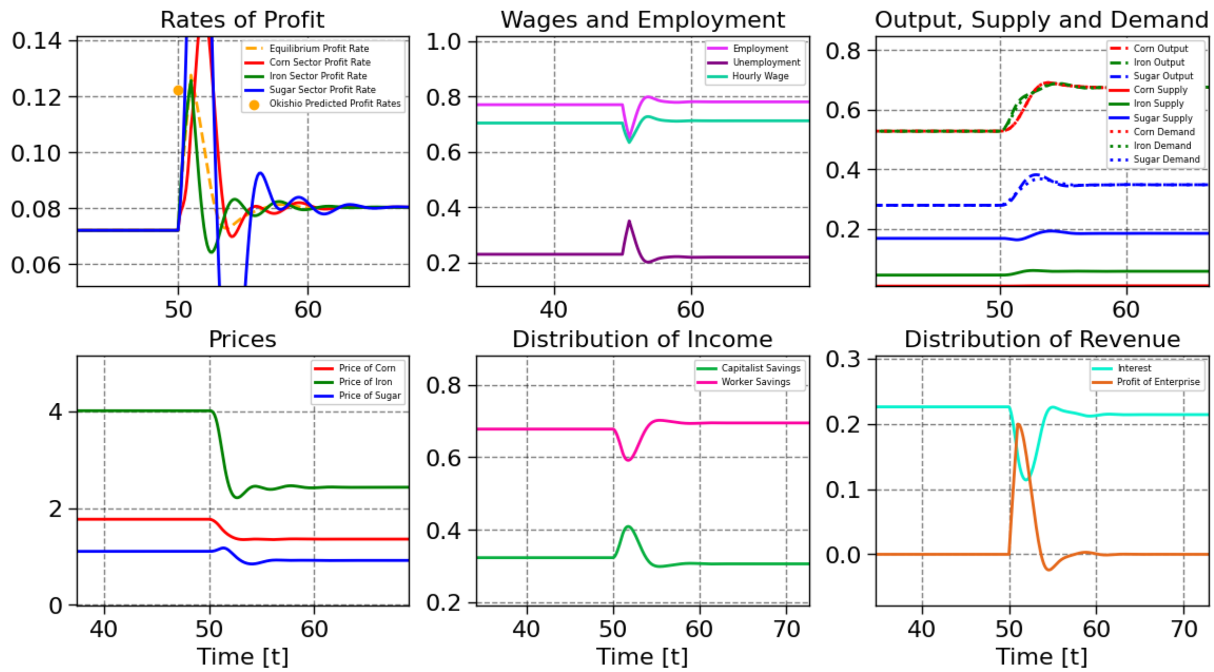


Figure 4: Basic dynamics of a pure labor saving change being suddenly introduced.

technological change (i.e. not capital using) tends to be the wildcard of the bunch. Profit rates can land above or below what they were depending on the industry they occur in, with equal consistency, as figure 5a shows.

The long term result of applying many pure LS shocks is significantly less mysterious, however. Observe that with no accompanying changes to the capital goods requirements, we have the limit

$$\lim_{\mathbf{l}\to\mathbf{0}} A + \mathbf{b}\mathbf{l}^T = A + \mathbf{b}\mathbf{0}^T = A \tag{22}$$

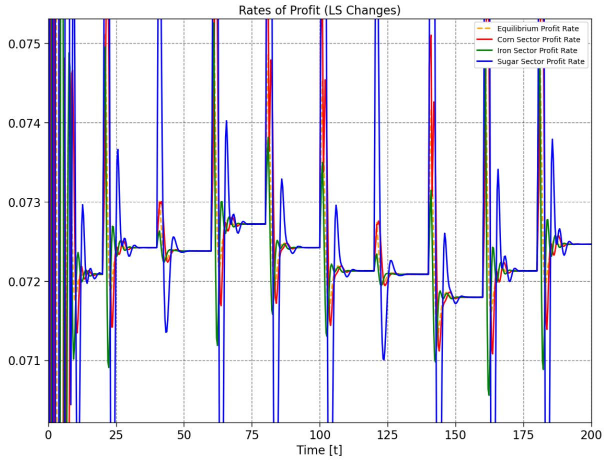
As the living labor vector approaches zero, the augmented requirements matrix approaches the basic input/output matrix A, whose spectral radius is inversely related to the maximal rate of profit plus one, i.e. the rate of profit which would prevail if wages are zero. Note that this happens regardless of whether wages actually go to zero, which by inspection of equation 6 they generally do not! Figure 6 depicts this. Interestingly, it takes quite a bit of time for the effects of this approach to be felt. The simulation depicted has 15% labor reductions being applied every 10 periods, it takes the system almost 500 cycles to even begin trending upwards, and 1700 cycles to finally reach it.

Now let us finally turn towards *capital using* labor saving technological change. Firstly, not all CULS change is cost-reducing, and so it is necessary to explain how I am implementing these techhnological shocks. Let $\beta > 0$ denote the proportion by which we wish to shrink the living labor requirements for a particular sector, i (in the application, this is denoted β_T). The wages needed prior to the technical changes amount to wl_i, while the cost of the capital goods required is $\mathbf{a}_i \cdot \mathbf{p}$, where $\mathbf{a}_i$ is the i^{th} column of A. We will seek an $\alpha > 0$ to scale the vector $\mathbf{a}_i$ by uniformally while maintaining the condition that the new cost is lower than the old cost. Thus we require

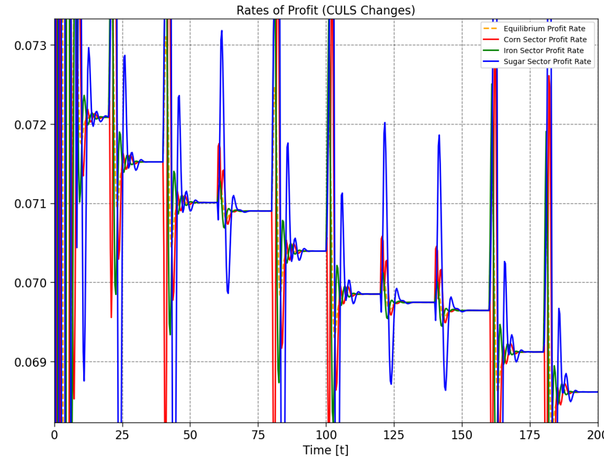
$$(1+\alpha)\mathbf{a}_i \cdot \mathbf{p} + (1-\beta)wl_i < \mathbf{a}_i \cdot \mathbf{p} + wl_i \tag{23}$$

Rearranging the terms yields

$$\frac{\beta}{\alpha} > \frac{\mathbf{a}_i \cdot \mathbf{p}}{wl_i} \tag{24}$$



(a) Applying small purely labor saving technological changes appears to not break any particular way for the rate of profit.



(b) Applying small *capital using* labor saving technological changes distinctly induces a fall in the rate of profit.

Figure 5: Effects of randomly induced labor saving (LS) and capital-using (CULS) technological changes every 20 periods. $\beta_T = 0.03$, $\epsilon_T = 0.01$

Suppose we are given some small factor $\epsilon > 0$ in addition to β. To decide on an α, we compute the right-hand side of 24 (call this the *cost-ratio*) and scale by $(1+\epsilon)$ to obtain the term

$$\gamma = \frac{\mathbf{a}_i \cdot \mathbf{p}}{wl_i}(1+\epsilon) \tag{25}$$

We then set this term equal to the ratio $\frac{\beta}{\alpha}$ and solve for α. To obtain CSLU changes is an identical procedure. The result is a ratio which is guaranteed to exceed the cost-ratio, thereby guaranteeing cost-reduction. The larger ϵ is (denoted ϵ_T in the application), the greater the cost-reduction, and the closer we get to the case of a purely labor reducing technological change.[14] The final adjustable parameter is D_T, which controls the frequency of the changes. Applying a reduction of $\beta_T = 0.03$, with $\epsilon_T = 0.01$ and $D_T = 20$, creates the effect seen in figure 5b. The results are shocking. The rate of profit *falls*, consistently and definitively.

This result is not unique to the parameters we have used, nor does it depend the sector receives the innovation. The reader is encouraged to generate some random parameters and try to convince themselves of this.[15] This is also not an artifact of the discrete nature in which we are applying the changes. To prove this, we can change the Scenario from Fits and Starts to Continuous to simulate the effects of continuous technical change of any kind. The way we accomplish this within the simulation is as follows. At the start of a time step, we choose a random sector of the economy which will receive continuous innovation, store that, and set a flag for the changes to occur. Within the simulation, all quantities are assumed to be floating, including the coefficiencts of the matrix A and the living labor vector $\mathbf{l}$ - they just generally are modelled as having a derivative of zero. As mentioned, BDF computes a trajectory by taking small steps and following the direction of slopes, which it continually recalculates. Within the calculation of the given changes for A and $\mathbf{l}$, the flag alters the behavior of the slope calculation to point towards changes to A and $\mathbf{l}$ which are along the lines of what we described for the discrete case. Figures 7 depict this for both LS and CULS

[14] Also note that $\epsilon = 0$ does *not* necessarily mean no changes to the A matrix. Additionally, values of ϵ greater than 1 turn the capital or labor increases into decreases. This allows one to model the forbidden CSLS technological changes, which have some interesting dynamics in their own right.

[15] The only condition is that the system needs to have settled into equilibrium already. To obtain a settled state prior to applying technological perturbations, use the 'Grab as initial' feature in the top toolbar. Type something like 100 or 200 into the entry widget, and then click the button to apply those settings as initial parameters and start with a system which is mostly 'at rest'.

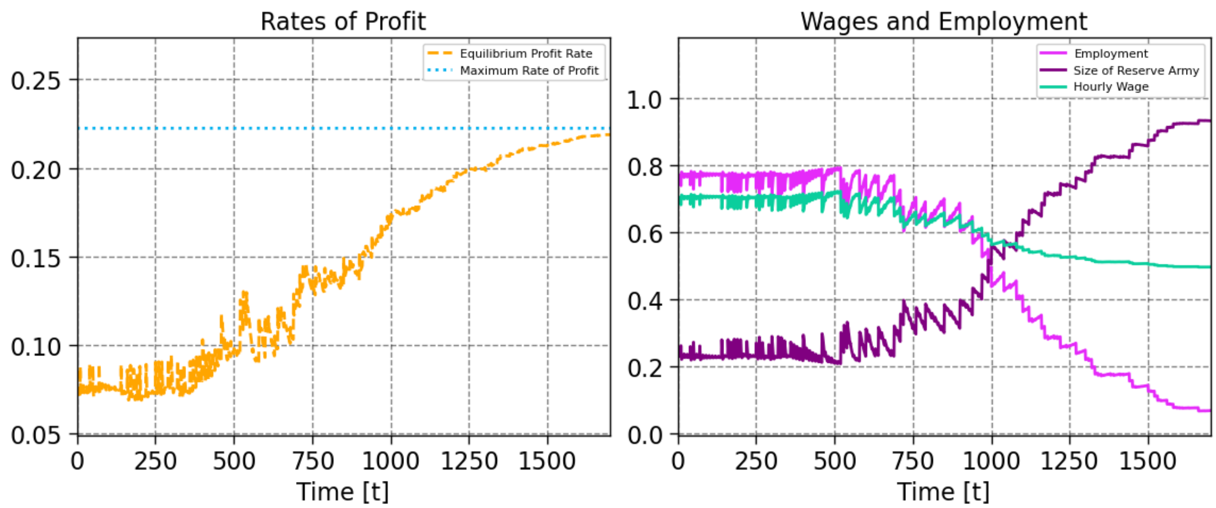


Figure 6: Eventually a sequence of purely labor saving innovations must tend the economy towards the maximum rate of profit, despite wages levelling out at a positive number.

change. Again we can note the striking way in which the equilibrium profit rate, shifting dynamically in the face of both technological and social changes, steers the sectoral rates.

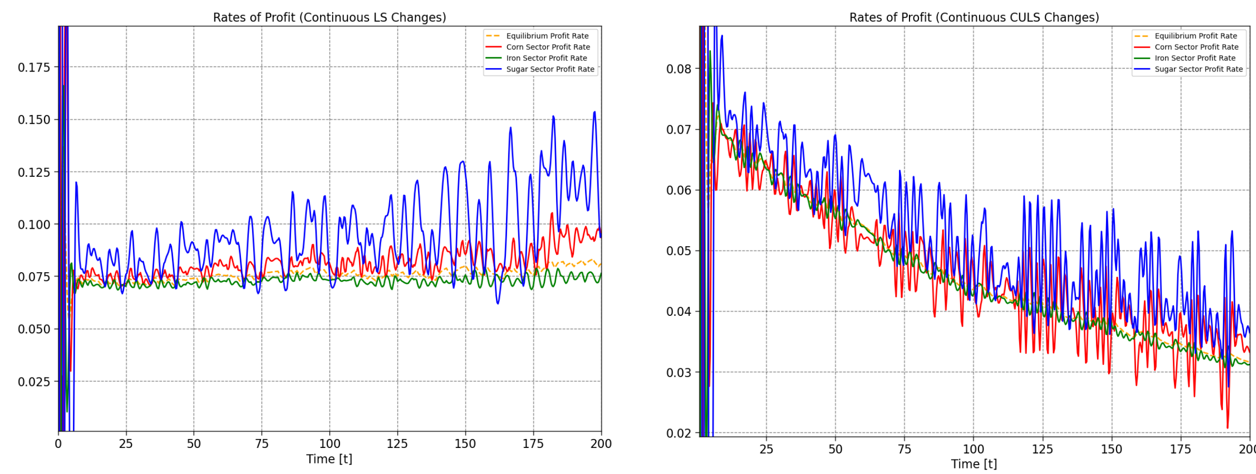
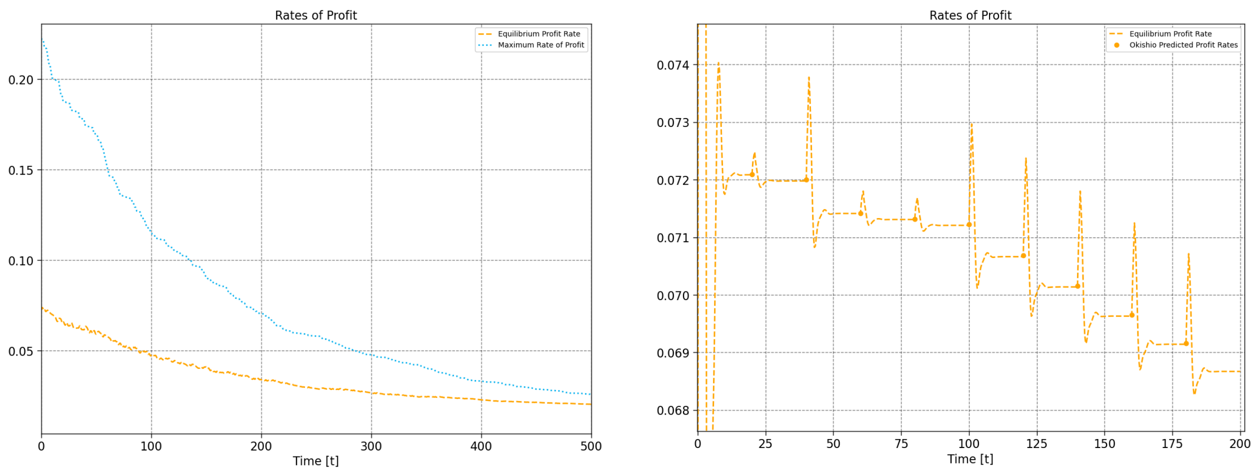


Figure 7: The observed results also apply in the case of continuous changes.

Another line of inquiry which followed after the results of Okishio from theorists such as Shaikh revolve around the maximal rate of profit, which as we mentioned earlier is the rate of profit which would prevail if the hourly wage were zero, i.e. if technical costs were all that existed.[18] Shaikh and others have observed that in the presence of CULS technological change, while Okishio dictates that the equilibrium rate of profit rises, the *maximal* rate of profit falls. This is because the maximal rate of profit is the quantity

$$\pi_{max} = \frac{1}{\rho(A)} - 1 \tag{26}$$

where $\rho(A)$ is the spectral radius of A. By the Perron-Frobenius theorem, an increase to the entries of any row of A has a strictly non-decreasing effect on the spectral radius, and consequently a strictly non-increasing effect on the maximal rate of profit. Shaikh argued that a falling maximal rate of profit would exert more and more ‘pressure’ on the actual rate, leading to increased volatility until a falling rate of profit ‘becomes inevitable’. As Roemer points out, this is a false conclusion to draw - one curve can fall while another rises continuously, while approaching each other asymptotically and never changing direction. Nonetheless, the claim of increasing volatility clearly cas some empirical validity here. Morever, the declining ceiling of a falling maximal rate of profit clearly exerts some broad influence on the dynamics here, as figure 8a demonstrates. Rather than approaching each other, however, we instead see both rates falling in tandem with one another. The maximal rate falls faster than the actual rate, whose the fall is muffled by the effects of the labor saving component of the innovation, but not reversed, so that they eventually come to meet each other asymptotically.



(a) Rather than approaching each other from both directions, the maximal rate of profit and the actual equilibrium rate of profit both fall in tandem in response to CULS changes.

(b) The results of Okishio bear next to no relevance towards the actual observed dynamics.

Figure 8: The observed results also apply in the case of continuous changes.

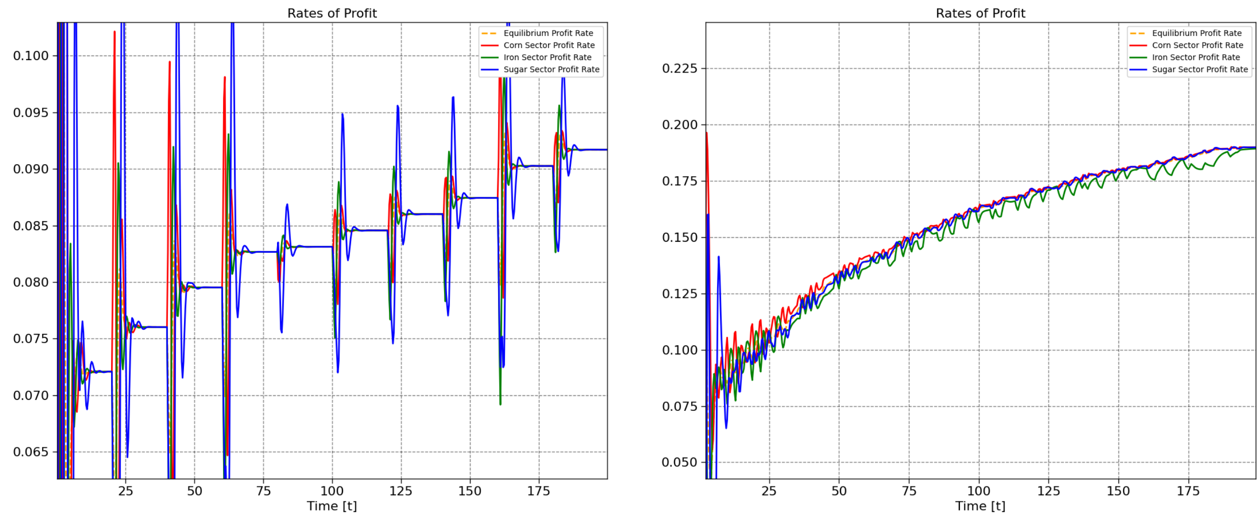
Figure 8b compares the profit rate changes as predicted by Okishio’s Theorem with a system undergoing discrete CULS technological shocks. We see that while the predictions are accurate for the immediate moment, they are otherwise completely irrelevant to what is observed. These findings urge us to dismiss Okishio’s theorem as having anything at all to say about more complex models of capitalist systems (or, for that matter, real ones). It should be noted that high enough values of ϵ_T will eventually cause our observations of a falling rate of profit to disappear. As we mentioned, higher values of this number correspond with smaller increases in the capital use. Clearly there is some minimal increase in the constant capital requirements relative to the decrease in labor requirements which enable the rate of profit to fall. However, as long as the pair (β_T, ϵ_T) falls within those requirements bounds, whatever they are, the fall is absolute, and the rate of profit is strictly decreasing.

4.1.2 Capital Saving Technical Change

Figure 9 depicts the general effects of introducing CSLU changes to an economy. In general, we witness a rising rate of profit in both cases, with both continuous and discrete applications of the change. The effects of introducing purely capital saving changes to the economy are no different in this regard.

Here, the maximal rate of profit blasts off to infinity as the requirements matrix A tends toward zero. However, the rising profit rate would seem to be bounded asymptotically by a ceiling of some sort. Why is this? Again consider the augmented requirements matrix $A + \mathbf{b}\mathbf{l}^T$. In the case of either CS or CSLU changes, the A term tends to zero in all entries. But the outer product $\mathbf{b}\mathbf{l}^T$ only gets bigger. As profitability increases, so does output, and with it increasing employment, driving up the total real wage $\mathbf{b}$. At best, $\mathbf{l}$ does not change at all. In effect, just as the maximal rate of profit shifts towards $\frac{1}{\rho(A)} - 1$ in the case of $\mathbf{b}\mathbf{l}^T$

going to zero, it shifts towards $\frac{1}{\rho(\mathbf{b}\mathbf{l}^T)} - 1$ in the case of A going to zero.[16]



(a) CSLU changes in fits and starts

(b) Continuous CSLU changes

Figure 9: The effects of introducing CS and CSLU changes to an economy have a generally positive effect on the rate of profit. (Here $\beta_T = 0.03$, $\epsilon_T = 0.01$)

4.1.3 Summary of Findings

Figure 10 summarizes our empirical findings for the four categories of technological change. We summarize these findings as follows. Pure labor saving technological change has the effect of a short-term immediate increase of the rate of profit, but the long-term equilibrium effect is generally neutral. The result can sometimes be an increase and sometimes be a decrease depending on the sector, but typically these effects are very small, and the economy returns to the same profitability as before. As the economy enters extreme levels of automation, this lack of bias vanishes, the system enters a 'breakdown' state and the rate of profit begins to rapidly climb towards the a maximum given by the non-labor requirements. Both purely capital saving and capital saving labor using technological change are both observed to have a definite long-term increasing effect on the rate of profit. The only qualitative differences between them are that pure CS changes tend to cause more volatility than CSLU changes, and that pure CS changes have an immediate positive effect on the rate of profit, whereas CSLU changes have an immediate negative effect on it. The reason for this immediate negative effect is actually quite simple: capital savings will increase the rate of profit in the sector which implements them, but then the ensuing spike in employment and thus wages will negatively impact *all* other sectors of the economy.

Finally, and most importantly, CULS changes are observed to have the effect of an immediate increase to the rate of profit (at least for the sectors experiencing the innovation), and a *long-term decreasing* effect on it. Within our model then, Marx's theory of a technologically induced falling rate of profit appears to be alive and well. These observations, while not yet formally proven, have held up to exhaustive empirical testing. No exceptions to the rule have been found, beyond what has already been discussed. We now proceed to analyze these findings.

4.2 Analysis

While I do not yet have a general theorem for when the rate of profit should be expected to fall, I do have a set of results and empirical observations which I believe combine to tell a compelling story. We approach this question in a layered manner.

[16] Recall that setting $\epsilon_T > 1$ with either CULS or CSLU changes selected allows one to consider CSLS change. In this case, the equilibrium rate of profit will always equal the maximal rate, with both numbers climbing without bound.

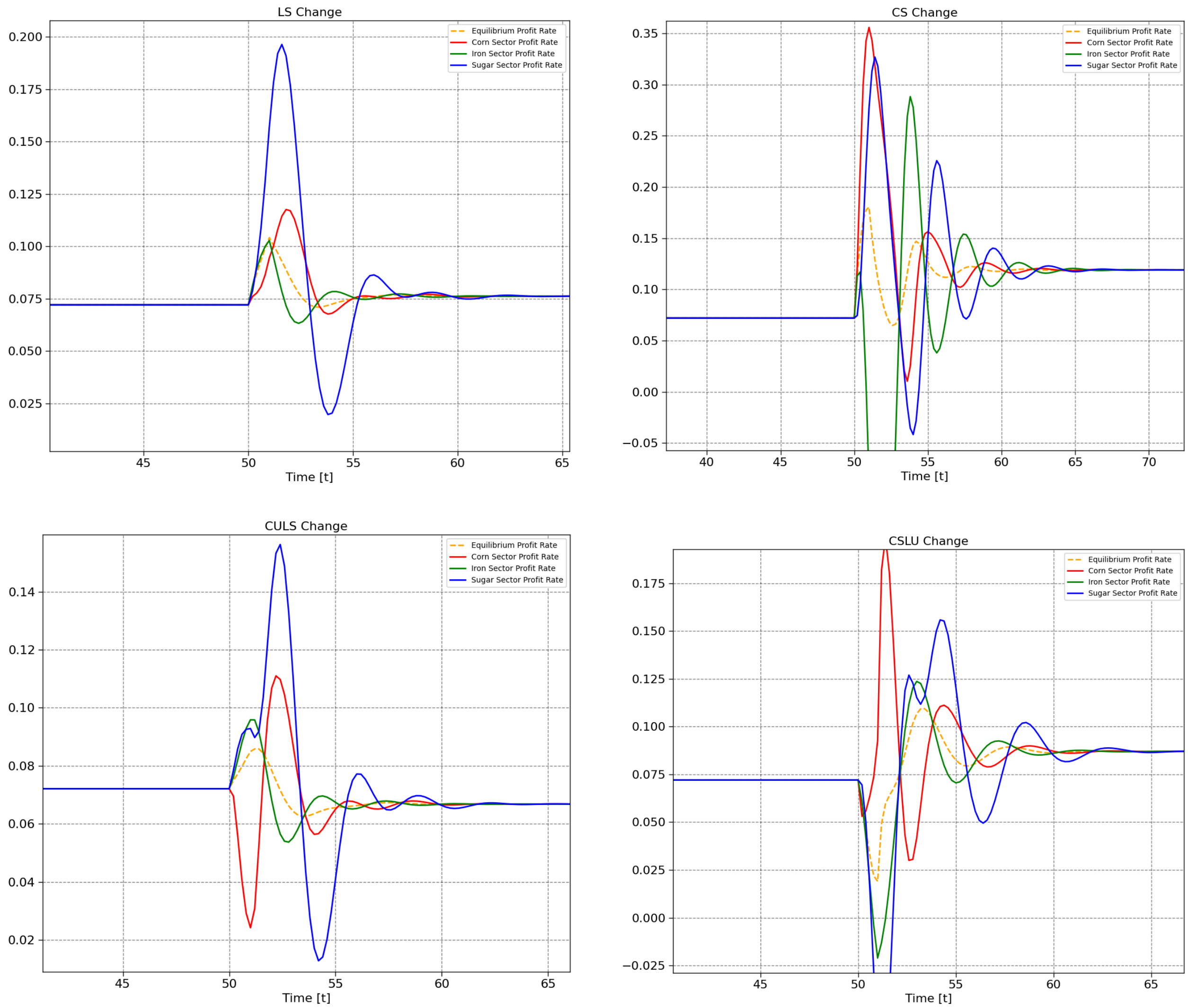


Figure 10: The effects of all four kinds of technological change. (Here $\beta_T = 0.3$, $\epsilon_T = 0.01$)

4.2.1 Distribution of Income

At the outermost layer we have the financial mechanisms - the interest rate, the money supply, and the distribution of income. When labor saving innovation is introduced into an economy, the employed population contracts, and wages fall. This results in a contraction of the overall worker savings m_w. But by identity 18, the share of money is zero-sum between the workers and the capitalists. Any amount of money leaving the worker's savings must enter into the savings of the capitalists, causing the interest rate to fall. In the absence of further technological innovation or worker immiseration, total profits of enterprise will quickly return to zero, and capitalists will return to being entirely reliant on interest returns for their income.

A shift in the money supply in favor of the capitalists results in a shift in the entire demand structure of the economy, away from labor goods (e.g. corn in our toy model) and towards luxury goods (e.g. sugar). As employment begins to ramp up again, demand shifts the other way. The oscillation of purchasing power between the workers and the capitalists creates reverberations in the composition of the net product. A chaotic process of resolving disequilibrium begins anew, in which the economy adjusts and readjusts according to the heterogeneous demand between the workers and capitalists ($\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$). When equilibrium is reasserted, one only needs to only look the total employment or the total worker savings to decide which

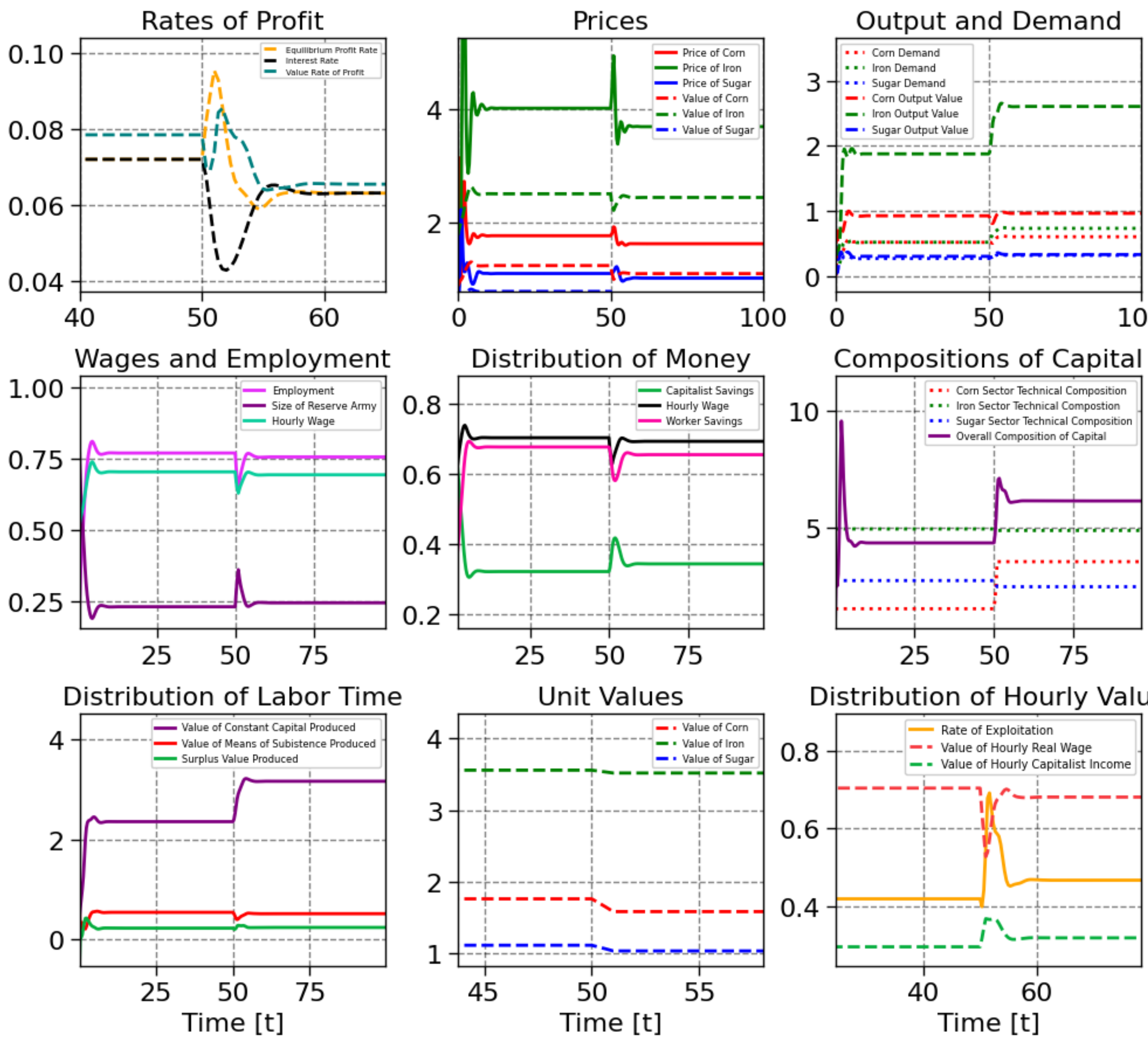


Figure 11: Noteworthy dynamics of a CULS perturbation with $\beta_T = 0.5$, $\epsilon_T = 0.01$, with innovation in the corn sector.

class the economy adjusted it's output bias toward. If total employment is higher than it was before, or equivalently if the total stock of worker savings is larger, then the economy will have shifted towards in favor of labor goods. Conversely, if total employment settles at a level lower than it was before, or equivalently the money stock has managed to stay permanently skewed towards the capitalists, then the economy will have shifted gear in the other direction towards the production of luxury goods for the capitalists.

The interest rate inversely reflects both of these scenarios. A more worker-oriented consumption economy means more of the total money supply must circulate in the service of that. Money will have become more precious for the capitalists, meaning that the interest rate will have risen higher than it was prior, and with it the rate of profit (which, again, is equal to the interest rate in equilibrium). Conversely if a more luxury-goods oriented economy emerges, this must be accompanied by a concentration of money in the hands of the capitalists, meaning a lower interest rate and a lower profit rate. Ironically, a more worker-oriented final output corresponds to a lower profit rate for the capitalists, and vice versa. These dynamics and more are all depicted in figure 11.

These observations are equally valid in reverse in the case of either CSLU or CS technological change. Capital savings boost profits of enterprise without the laying off of workers. This causes in both cases a surge of employment, inducing a net transfer of the total money stock from the hands of the capitalists into the hands of workers. This makes money more scarce from the standpoint of lending, and leading to a spike in interest rates.

One might be tempted to call story finished here, and argue that since oscillations dampen as a system returns to equilibrium, the interest rate will come to rest at a lower number than it had been prior to the change in technique. To frame matters more oppositionally, we could say that when the industrial capitalists implement their changes, they assert power for themselves at the cost of the financial capitalists, but since these gains are ephemeral, and the power lost by finance is never fully recovered, the long term effect this has on the capitalist system to the detriment of the class as a whole. This is a fun story, but fails to fully explain the phenomena, since if it did then it would also have to apply equally to the case of *pure* labor saving innovation. As we saw, at least until the economy is at a very high degree or automation, pure labor saving technology has a more or less neutral effect on the profit rate. There must therefore be something specific to *capital using* labor saving technological changes which causes the phenomenon.

The following lemma confirms the assertions we made that the final level of employment and/or the final level of worker savings completely determines whether the profit rate rises or falls after technical changes.

Lemma 4.1. *Suppose that a system in equilibrium is suddenly hit with a technological change (i.e. any change to the A matrix and/or the living labor vector $\mathbf{l}$), and assume that the system eventually returns to an equilibrium after this change. For parameters p, we let p_0^* denote the equilibrium value for the parameter prior to the change in technique, and p_1^* denote the equilibrium value for it after. Let $E_i^* = \mathbf{l}_i \cdot \mathbf{q}_i^*$ denote the equilibrium level of employment. Then, assuming that the total circulating money in the economy remains fixed at M, we have*

$$r_1^* < r_0^* \Leftrightarrow m_{w0}^* > m_{w1}^* \Leftrightarrow \frac{E_1^*}{E_0^*} < \left(\frac{L - E_1^*}{L - E_0^*}\right)^{\eta_w} \tag{27}$$

Proof. Note that equation 10 depends only on the capitalist savings m_c and can be solved separably to yield the closed form equation:

$$r(t) = \frac{r(0)m_c(0)^{\eta_r}}{m_c(t)^{\eta_r}} = \frac{r(0)(M - m_w(0))^{\eta_r}}{(M - m_w(t))^{\eta_r}} \tag{28}$$

Thus the equilibrium interest rate prior to change in technique is a function of the equilibrium worker savings:

$$r_0^* = \frac{r(0)(M - m_w(0))^{\eta_r}}{(M - m_{w0}^*)^{\eta_r}} \tag{29}$$

$$r_1^* = \frac{r_0^*(M - m_{w0}^*)^{\eta_r}}{(M - m_{w1}^*)^{\eta_r}} = \frac{r(0)(M - m_w(0))^{\eta_r}}{(M - m_{w1}^*)^{\eta_r}} \tag{30}$$

Where the second equation for r_1^* follows from substituting the first equation for r_0^* and cancelling like terms. Substituting these equations into the inequality $r_0^* > r_1^*$ and simplifying yields

$$r_0^* > r_1^* \Leftrightarrow \frac{1}{(M - m_{w0}^*)^{\eta_r}} > \frac{1}{(M - m_{w1}^*)^{\eta_r}} \Leftrightarrow m_{w0}^* > m_{w1}^* \tag{31}$$

Next, we note that in equilibrium, workers spend what they earn, i.e. $E_i^* = (\mathbf{l}_i \cdot \mathbf{q}_i^*) = \alpha_w m_{wi}^*$ for $i = 1, 2$. This is seen plainly by setting $\frac{dm_w}{dt} = 0$ in equation 6. Substituting these into the inequality $m_{w0}^* > m_{w1}^*$ immediately gives us

$$E_0^* w_0^* > E_1^* w_1^* \Leftrightarrow \frac{E_1^*}{E_0^*} < \frac{w_1^*}{w_0^*} \tag{32}$$

Similarly to $r(t)$, equation 6 can be solved separably under the assumption that both L and $\mathbf{l}$ is unchanging (in terms of calculating long-term equilibria before and after a single instantaneous change in $\mathbf{l}$, is the case), to yield the twin equilibrium equations

$$w_0^* = \frac{w(0)(L - E(0))^{\eta_w}}{(L - E_0^*)^{\eta_w}} \qquad w_1^* = \frac{w(0)(L - E(0))^{\eta_w}}{(L - E_1^*)^{\eta_w}} \tag{33}$$

Substituting these two equations into $\frac{w_1^*}{w_0^*}$ confirms the second inequality. □

In the case of $\eta_w = 1$, equation 27 amounts to the assertion that $E_1^* < E_0^*$. For general η_w, the conditions are more complicated, but it is never as cut and dry as tuning a particular η_w to ensure that the rate of profit does not fall, because η_w also conditions the initial equilibrium unemployment prior to the shock. In my extensive empirical testing I have been unable to witness the post-shock interest rate settling above what it was prior, for any setting of η_w, even very extreme ones. While this lemma represents progress towards resolution of the mystery, it does not explain *why* final employment levels tend to come back to rest at levels low enough to consistently guarantee a fall in the rate of profit. Thus we descend further to the second layer of our analysis.

4.2.2 Output and Employment

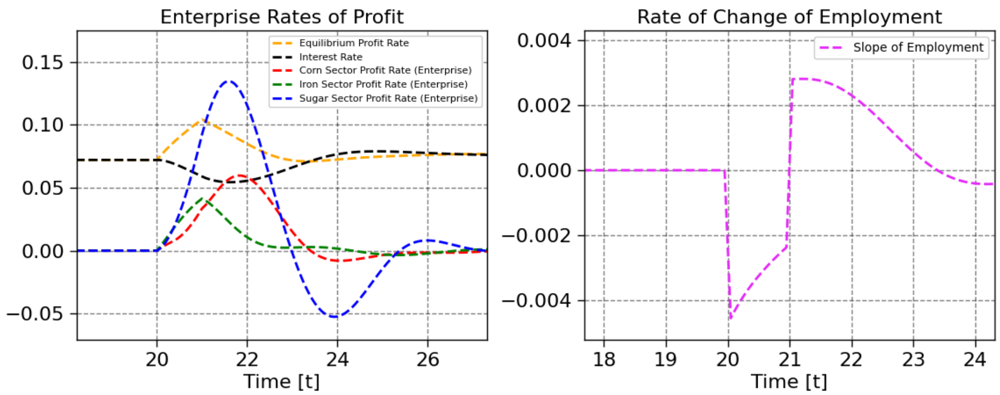
Let us consider a phase of this process in which employment and wages are both rising. As this process goes on, the general trajectory of all costs will be rising in turn. Demand for luxury goods already exists from the transfer of money savings to the capitalists, and demand for wage goods is rising with employment. To the extent that both of these rely on capital goods, demand for those will be rising as well. But observe that this upsurge in demand will generally be biased towards the capital goods above all, not just because all other demand relies on them but because *the technological changes increased the degree to which this is true*. This increased reliance on capital goods drains supply faster, increasing their prices faster, raising profitabiltiy higher, and generally biasing the economic expansion in their favor. The exact way in which this bias manifests itself numerically will be the subject of layer three of our analysis. For now, it will suffice to simply make the observation informally.

The period of accumulation, which is the period in which the overall profit of enterprise is disturbed from its rest state of zero, can be understood as a 'race' between the various different sectors. All sectors eventually begin attempting to expand production simultaneously, draining resources from one another at varying speeds according to the new cost realities brought about by the change in technique. The resulting supply disruptions creates profit rate heterogeneity, and with it heterogeneity of the sectoral rates of accumulation. This race continues until economic capacity is reached, represented as the point at which further expansion of production is no longer profitable. What we have have argued is that the particular character of CULS technological change rigs this race in favor of the capital goods sectors, and in particular in the favor of those sectors which have had demand increased by the new technique introduced.

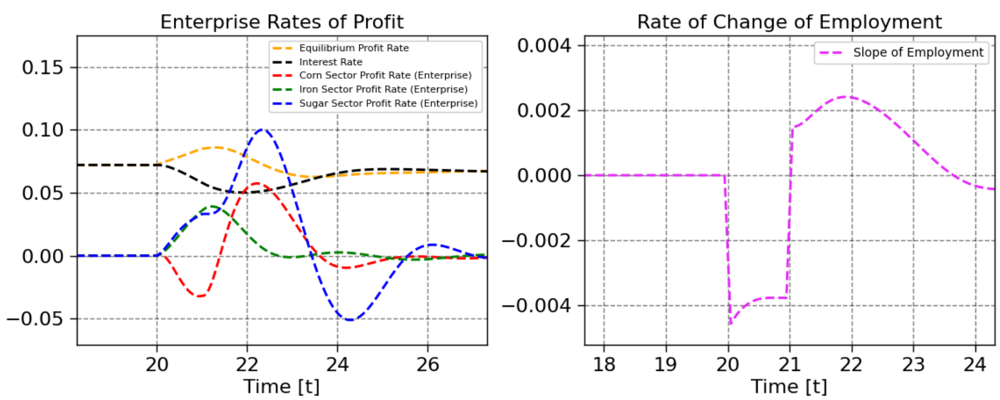
On the other hand, the increased materials requirements stunt the profitability phases for all sectors relying on the capital goods which are in heightened demand. This stunted phase of enterprise profitability also stunts the degree to which employment can recover. Figure 12 shows a 30% reduction in labor requirements for iron being applied, first in the case of pure labor saving and second at the cost of increase capital requirements for the production of iron. In the former case, the profitability phase is clearly sharper and more pronounced. This results in a lower peak increase in employment. On the other hand, the degree to which employment decreases at the start of the event is *unchanged* by the increased capital use. The number of layoffs depends only on the degree of labor saving, β_T. If 30% less workers are suddenly not needed for the current level of output, 30% of workers will be laid off immediately, irrespective of whether or not the capital requirements increase.

To summarize, increased capital requirements serve to stunt the rehiring phase of a technological shock while having little to no effect whatsoever on the firing phase. Thus the total change in employment over course of the event is negative. Hence, by lemma 4.1, the rate of profit falls.

Just as was the case for layer 1 of the analysis, all of the observations just made can be applied in an inverse form to the case of both CS and CSLU changes. Lower capital goods requirements makes labor



(a) Equal capital requirements before and after the shock allow employment levels to recover unimpeded.



(b) Increase capital requirements have a blunting effect on the profit rates, which stunt employment leaving it short of where it was originally.

Figure 12: Net change in employment is the integral of the change from the moment the shock occurs until the moment it settles. With pure labor saving innovation, this net change is roughly zero. With the simultaneous introduction of additional capital requirements, profitability is blunted, resulting in a net change which is generally negative.

goods appear scarce relative to them, biasing profit rates in their favor and leading towards a fundamentally a more labor intensive, less capital intensive society as equilibrium is reasserted.

4.2.3 Labor Time Accounting

We are beginning to see an interplay between the financial mechanisms and the new objective techniques of production. The profitability signals which guide the system into a new equilibrium are constrained by these techniques, which we have argued are biased in favor of the production of capital goods at the cost of all else.

We must return to the question of exactly what we meant when we claimed that CULS change in technique resulted in a productive system which is more biased towards capital goods production. It is tempting to conjecture this bias in classical value theoretic terms, and assert that the true cause of this falling rate of profit is the rise in the day's necessary labor at the cost of the surplus part. [17]. This is indeed often what we see (usually even).

However, it is not a necessary condition. Figure 13 depicts a CULS change in which the rate of profit falls despite a classical value-theoretic analysis concluding precisely the opposite. The composition of capital

[17] The classical Marxist categories of total constant capital, total variable capital and total surplus value C, V and S are

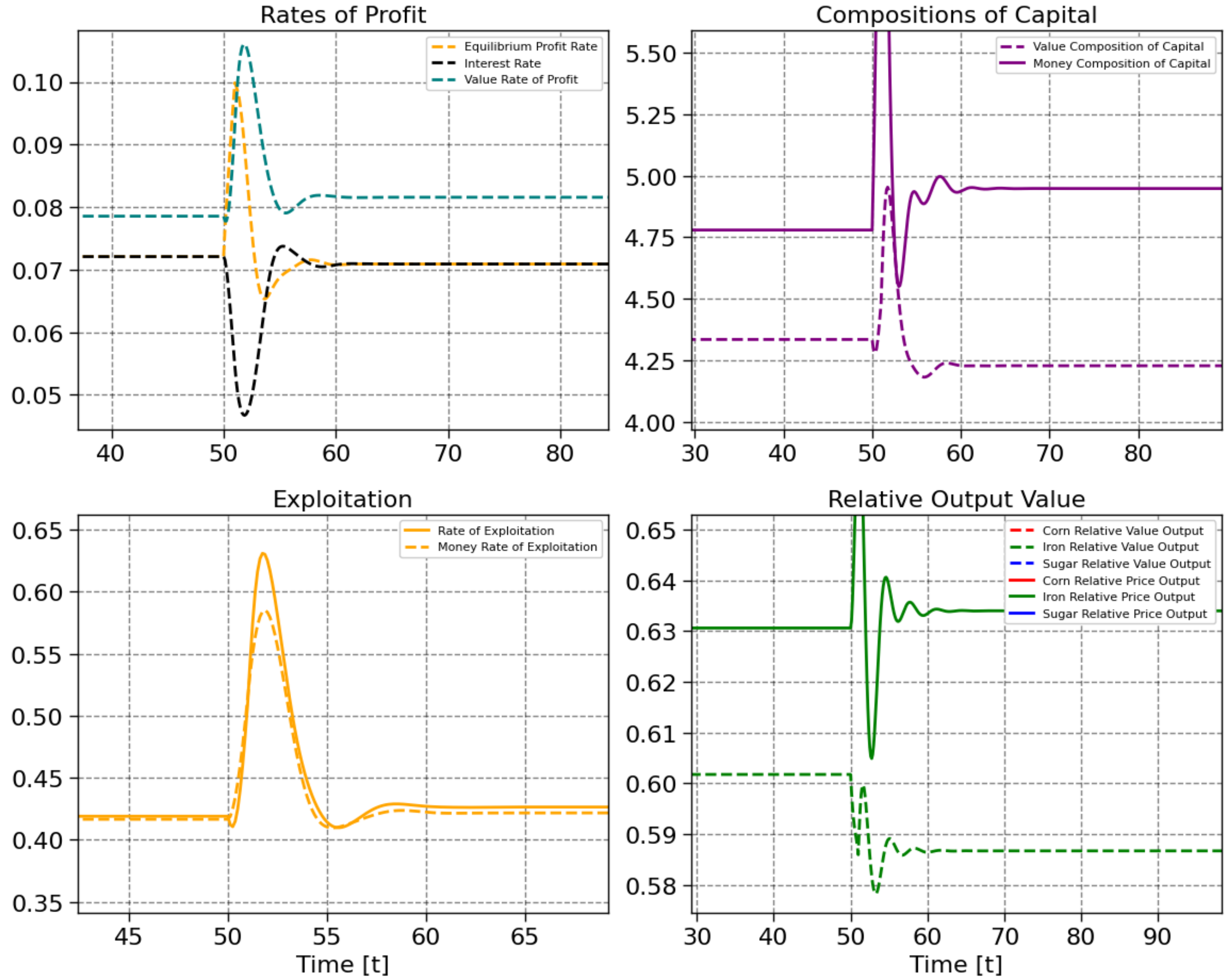


Figure 13: Observing the classical vs super-integrated labor time accounting systems in response to CULS changes with $\beta_T = 0.4$, $\epsilon_T = 0.8$. The value system is incongruous with the price system.

falls, as does the total output value of iron. We see the value rate of profit and the actual rate of profit diverge as a result of this.

Also depicted in figure 13 is the composition of *money capital advanced*, i.e. the quantity

$$\hat{k} = \frac{(A^T \mathbf{p}) \cdot \mathbf{q}}{wE} \tag{37}$$

Additionally, figure 13 depicts a 'money rate of exploitation', which is defined

$$\hat{e} = \frac{\alpha_c m_c}{\alpha_w m_w} \tag{38}$$

operationalized within our model with the equations

$$V = \mathbf{b} \cdot \lambda = \frac{\alpha_w m_w}{\mathbf{p} \cdot \underline{\mathbf{b}}} (\underline{\mathbf{b}} \cdot \lambda) \tag{34}$$

$$S = \mathbf{q} \cdot \lambda - V \tag{35}$$

$$C = (A\mathbf{q}) \cdot \lambda \tag{36}$$

where λ is the vector of unit values obtained by solving the equation $\lambda = A^T \lambda + \mathbf{l}$. From here, the value composition of capital is defined as $k = \frac{C}{V}$, the rate of exploitation by $e = \frac{S}{V}$, and the value rate of profit as $\pi_v = \frac{S}{C+V} = \frac{e}{k+1}$

This name is not given for any reason other than that it tends to closely track with the actual value rate of exploitation. Multiplying r^* by $1 = \frac{\mathbf{m}^* \cdot \mathbf{q}^*}{\mathbf{m}^* \cdot \mathbf{q}^*}$ and substituting identities 17, 19, and 20 gives

$$r^* = \frac{r^*(\mathbf{m}^* \cdot \mathbf{q}^*)}{\mathbf{m}^* \cdot \mathbf{q}^*} = \frac{\alpha_c m_c^*}{(A^T \mathbf{p}^*) \cdot \mathbf{q}^* + w^* E^*} = \frac{\alpha_c m_c^*}{(A^T \mathbf{p}^*) \cdot \mathbf{q}^* + \alpha_w m_w^*} \tag{39}$$

Dividing both numerator and denominator by $\alpha_w m_w^*$ yields

$$r^* = \frac{\hat{e}}{\hat{k} + 1} \tag{40}$$

which is effectively a 'money' analog of Marx's famous value rate of profit equation. By lemma 4.1, $r_1^* < r_0^*$ iff $m_{w1}^* < m_{w0}^*$. Clearly by the zero sum nature of the money supply between workers and capitalists this itself is true iff $\hat{e}_1^* > \hat{e}_0^*$. But then by observation of 40 it must follows that this $r_1^* < r_0^*$ iff $\hat{k}_1^* > \hat{k}_0^*$. It is *this* composition of capital rising which causes the rate of profit to fall, *not* the value composition of capital. The capital goods bias manifests first and foremost as a bias in the *distribution of money capital*.

We have come full circle back around to the first layer of analysis. The key thing really is the money system. However, we are no longer considering the money system in relation to interest rates, but rather in relation to what the distribution of money amounts to with respect to the distribution of labor, resources, and sectoral bias. The classical value system is a reliable measure of the distribution of labor. But the distribution of labor does not perfectly represent bias from the perspective of the capitalist's bottom line, which looks at the allocation of resources through the distorted lens of the money system. When the rate of profit falls, it has fallen not because the interest rate has fallen, but because a greater relative amount of the total money capital is tied up in the production of capital goods then in the production of consumption goods, *both* for workers *and* capitalists. Due to the equivalence between the total money capital invested in the production of goods for capitalists and the total real income capitalists receive, it follows that capitalists are receiving less income per dollar of capital investment. Hence a fall in the rate of profit. The changed interest rate merely reflects this shifting of priorities, which has been brought about by a shift in technology which relies more on capital goods and less on living labor.

We have verified empirically and justified logically Marx's fundamental correctness in his assertion that the rate of profit falls in response to CULS technical changes. However, we accomplished this under the assumption that the interest rate floats according to the distribution of income between workers and capitalists. In our modern world of debit cards and electronic bank accounts, this is not the case. To truly decide if his claims hold up in reality, we must consider the case where the interest rate is fixed and unchanging.

5 Technical Change with Fixed and Zero Interest Rates

5.1 Findings and Analysis

The interest rate can easily be fixed within our model by simply setting $\eta_r = 0$. Doing this and applying a CULS technical change yields predictable results, as figure 14 shows.

When enterprise profitability dries up, we return to equilibrium, where the interest rate is fully determinate of the rate of profit. Since this has not changed, neither has the rate of profit. The same will also be the case for any other category of technical changes.

However, this only demonstrates that the *equilibrium* rate of profit cannot change in response to technical changes. In the case of continuous technical changes, the system is never allowed to settle into equilibrium, and all bets are off. Figure 15 depicts the effects of continuous CULS changes with a fixed interest rate of 0.03. For clarity, we are only applying the changes to the iron sector. The effects are mostly the same when considering changes to randomly chosen sectors, depicted in figure 15. Here, we see some initial extreme variation due to the fact that differential levels of reliance on labor translate to differential profit spikes in response to the same percentage labor reduction. As the living labor as a whole approaches zero, these even out, and the behavior resolves into what we see in figure. The parameters are the same running example we have been using except that the system has been allowed to settle into approximate equilibrium prior to $t = 0$. Initially, profits of enterprise are zero, and the overall rate of profit equals the interest rate (0.03). When technological changes begin getting applied, the profit rate spikes immediately.

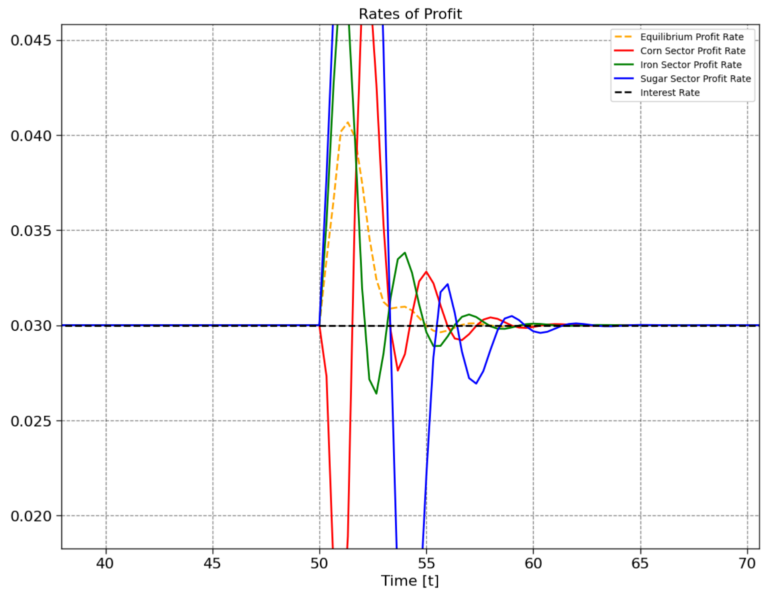


Figure 14: With interest rates fixed, there a reassertion of equilibrium ensures that there can be no changes to the rate of profit, regardless of the type of technical change.

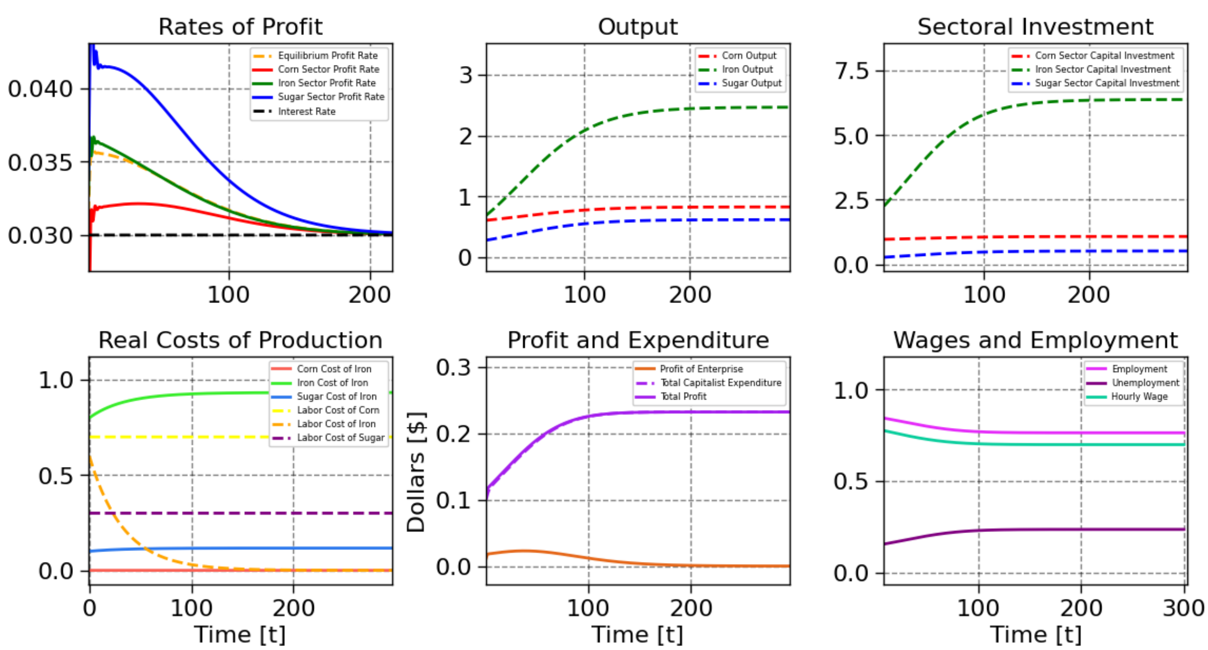


Figure 15: Effects of continuous CULS technical change with $\beta_T = 0.03$, $\epsilon_T = 0.01$. The rate of profit still falls. Starting parameters are our default example, but after 100 cycles (i.e. it has been given time to come 'to rest' before changes are applied.)

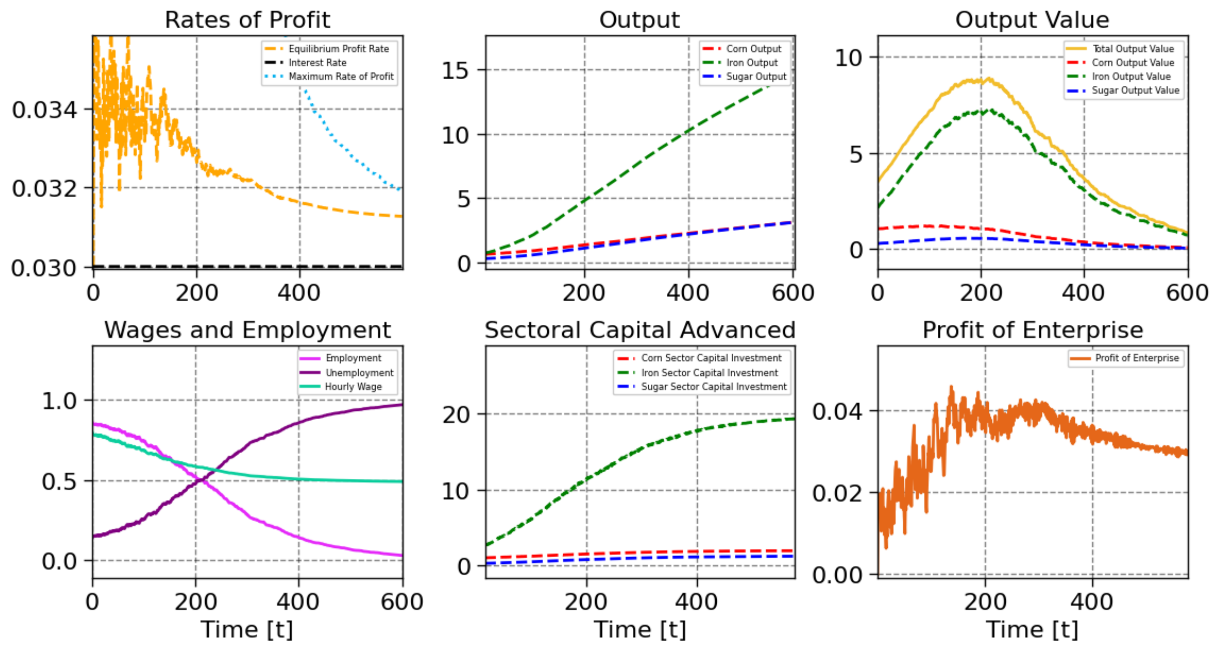


Figure 16: When changes are applied to all sectors randomly, the rate of profit falls but is asymptotically bounded above the interest rate.

Just as before, the rate of profit is unambiguously falling. The fixed interest rate acts as a floor which the falling profit rate descends towards. If the initial interest rate is zero, i.e. our system has no credit system to speak of, the effects are the same, since this is just a special case of the fixed interest rate model where the initial rate is zero. It is worth noting that the profit rate only approaches the interest rate in the case of changes to a single industry. If changes are widespread across all sectors, then the profit rate is still falling, but tends to asymptotically approach a minimal number which is still bounded above the interest rate, as depicted in figure 16. The reason this happens is that the changes applied to the requirements matrix A are dependent on the cost reducing capabilities of the living labor vector $\mathbf{l}$. As $\mathbf{l}$ approaches zero, shrinking the profit margin and thereby the maximum degree by which capital goods requirements can increase. The A matrix therefore converges as employment approaches zero, so that the rate of profit comes to meet a maximal rate given by A rather than the interest rate. It therefore appears to be the case that the maximal rate of profit will always tend to fall to a minimal level above whatever the fixed interest rate is. Further investigation is needed to fully understand this phenomenon.

The same effect is witnessed in the case of a fixed interest rate of zero, i.e. no credit system at all. Figure 17 depicts the effects of continuous technical changes to the iron sector specifically (for clarity) with $r(t) = 0$.

Why is this happening? One rather straightforward answer would be to point out that as these changes are applied, the money composition of capital approaches infinity, i.e. the labor costs represent a shrinking portion of overall costs. In lieu of this, a fixed percentage reduction in labor costs amounts to less and less in the way of super-profits, tending towards zero. However, there is quite a bit more we can say about what we are witnessing than just this.

The other plots of figures 17 and 16 tell a familiar story. As CULS changes are implemented, social production biases itself towards the production of capital goods, which may or may not manifest in form of an increase in the total value output of iron, but *always* manifests in the form of an increase to both the relative output of capital goods and an increase in the relative share of overall money capital invested towards the capital goods. This shift in the distribution of money capital towards capital goods exceeds the shift away from money capital invested in employing workers, resulting in a gradual fall in the dollars of profit income yielded per dollar of investment. Finally, we see in the Profit and Expenditure plots that equation 20 remains approximately true even outside of equilibrium. As long as it remains the case, even approximately, that capitalists 'earn what they spend', and as long as the money supply is fixed, a shift in the

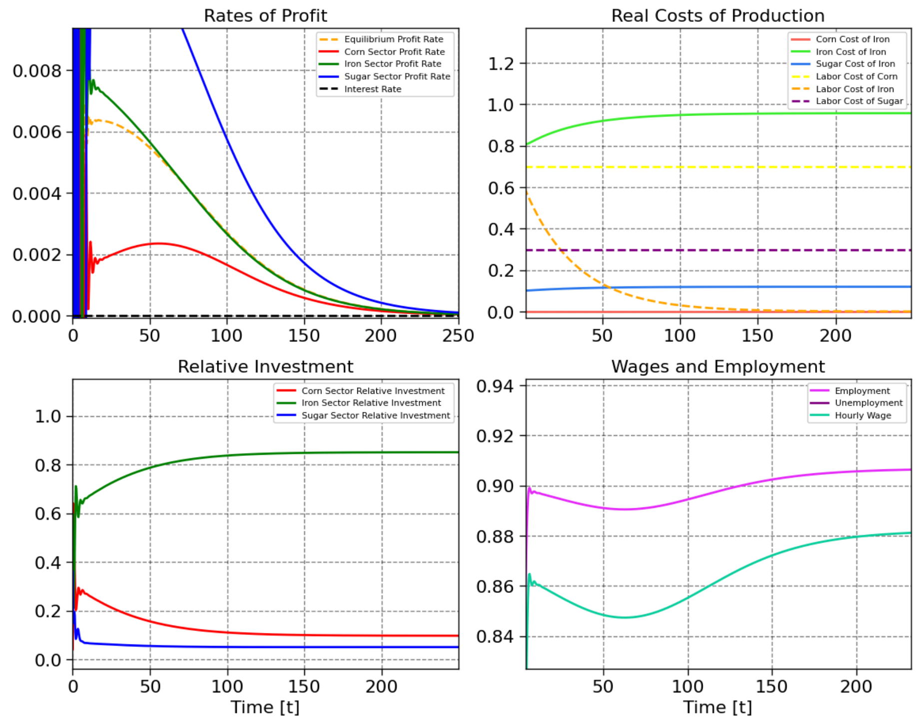


Figure 17: Effects of continuous CULS technical change to the iron sector specifically with $\beta_T = 0.03$, $\epsilon_T 0.01$ in the case of zero interest, i.e. $r(t) = 0$ for all t. The rate of profit still falls.

relative share of investment towards the capitalist goods which is not made up for by an equal shift towards the production of capitalist consumption goods must result in a fall in the total dollars of capitalist income earned per dollar of investment, i.e. a fall in the rate of profit. The rate of profit remains safeguarded from below by the interest rate, as a sort of ‘profit of last resort’, ensuring a minimal level of investment into the capitalist consumption sectors, but the extra profit of enterprise experiences diminishing returns for reasons which are both very similar to Marx’s argument and understandable via observations we already made in the case of discrete changes and a floating interest rate. The only necessary revision to Marx’s argument is that we must think not in terms of the distribution of labor, but rather in terms of the distribution of money capital investment. This is an echoe of the mismatch between prices and values known as the transformation problem.

If the money supply is not fixed, then anything short of *continuous* injections of new money directly into the pockets of the capitalists can possibly hope to be sufficient to counter these effects, for any discrete injection of money into the economy will cause an immediate readjustment of prices via inflation, leaving the *distribution* of money capital investment unaltered. [18]

Figure 18 depicts the results of continuous changes of all four kinds on the rate of profit, on an economy starting at rest. Labor saving innovation causes the profit rate to flies away from the interest rate and escape it’s gravitational pull. The reader can verify for themselves that the distribution of money capital investment shifts away from corn, stays mostly constant for iron, and increases for sugar, and additionally outputs for all commodities grow in a more or less balanced fashion. In the case of CSLU innovations, we see a similar gravitational pull towards the interest rate but from the other direction. We already observed why CSLU changes were initially harmful towards the rate of profit - the increased cost of labor in multiple sectors tends to outweight the cost reductions in the one sector receiving the changes. However, the gravitational pull is interesting, because it witnesses a symmetric law of diminishing ‘returns’. As the CSLU changes continue to be applied, the necessary condition of cost-reduction combined with the slowing logistical descent towards zero capital goods requirements reduces the degree to which labor use can be increased, slowing down it’s

[18] This may serve to explain the phenomenon we have seen referred to euphemistically by the federal reserve as ‘quantitative easing’.

negative effect, and allowing the interest rate to eventually dominate with its gravitational pull. Finally, pure capital saving technology initially avoids the negative effects of increased labor costs and allows the profit rate to initialize to a higher value than the interest rate. Aside from this though, changes of this type are a bit of a wildcard. The effects of continuous pure CS changes with a fixed interest rate vary wildly depending on the specific industry that the changes are applied to. In particular, if the changes are applied specifically to the corn sector, profit rate equalization is prevented entirely, with iron remaining at an interest rate permanently *lower* than the interest rate! A significant amount more work is required to fully understand these changes in the case of fixed interest rates, which are not the subject of this paper.

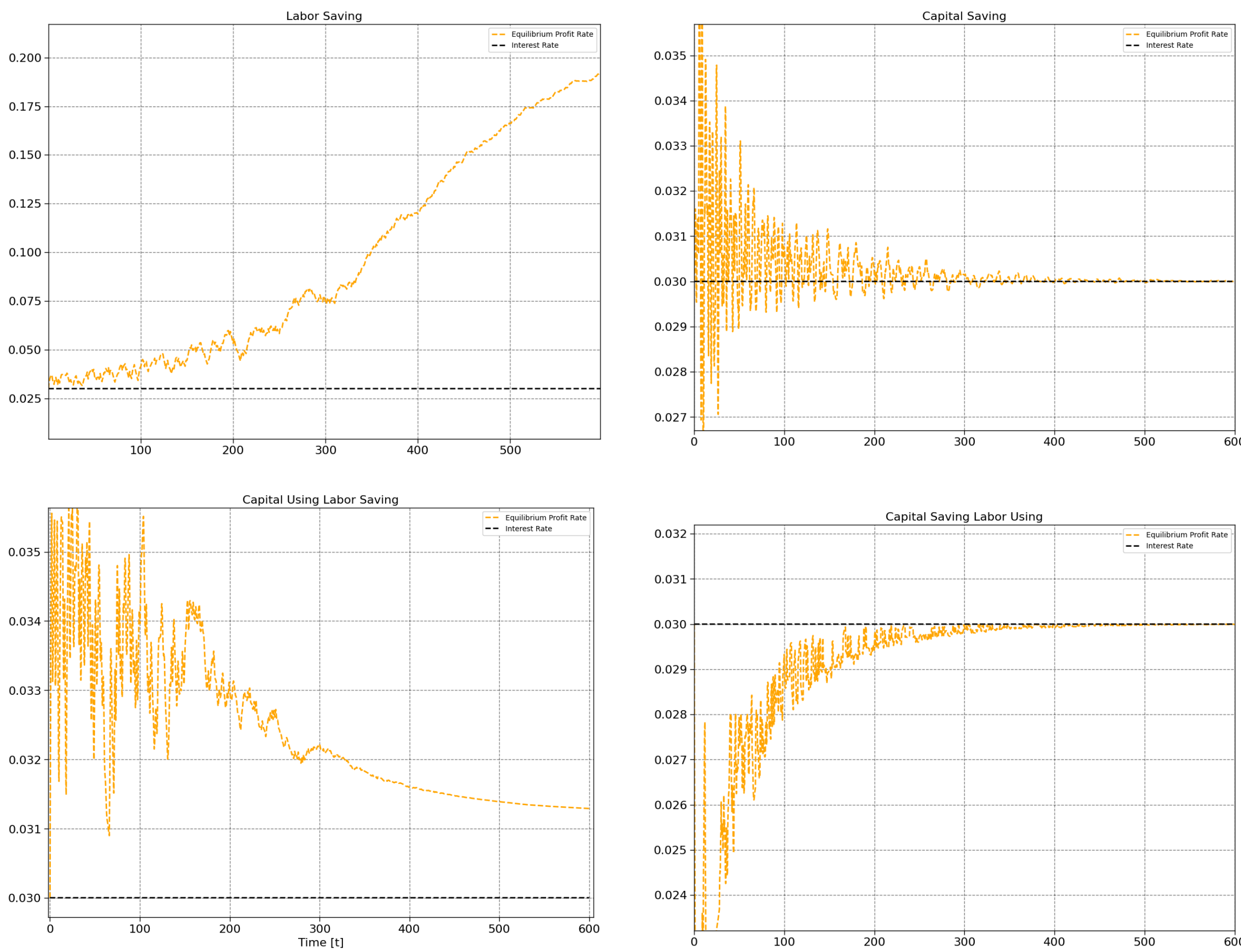


Figure 18: The effects of all four kinds of technological change in the case of fixed interest rates. (Here $\beta_T = 0.3$, $\epsilon_T = 0.01$)

As a final demonstration, figure 19 depicts the effects of the same continuous technical changes with a fixed interest rate but *with the real wage fixed*. We can therefore see that this completely negates our result - as Okishio predicts, the rate of profit continually rises. It is worth emphasizing that we go from figure ?? to figure 19 with no alterations other than a fixing of the real wage as its starting value. This one assumption makes a falling rate of profit impossible to see!

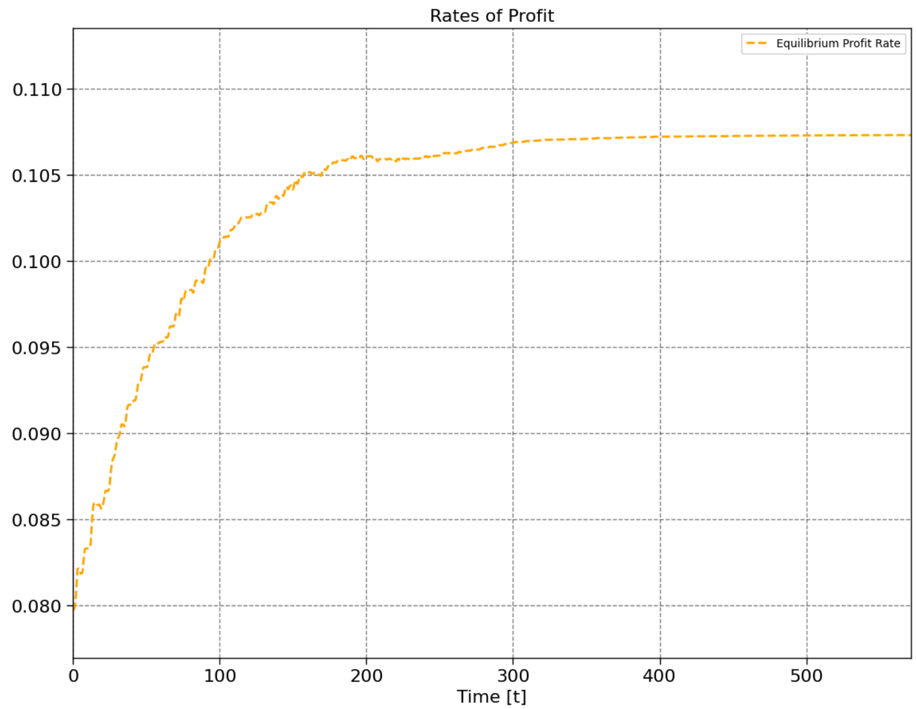


Figure 19: Our results are completely the opposite of what Okishio's theorem would predict. Parameters are saved under the title 'Continuous Okishio Fixed Interest Rates'

6 Conclusion and Further Research

We summarize our findings as follows. First, Okishio's theorem, while mathematically and even dynamically valid in it's own special context, must simply be discarded to irrelevancy in the context of more sophisticated dynamic models where money and credit dynamics are properly defined, prices are not continually renormalized in a discontinuous manner, and the real wage is allowed to float. The predicted profit rates of that model have seemingly no bearing on the dynamics of these systems, and in many cases completely contradict them.

Next, we find that within the rich dynamics of cross-dual models of classical graviation where all relevant quantities are allowed to float and adjust continuously in relation to one another, Marx's argument for a technologically induced rate of profit appears to be completely revitalized. Capital using, labor saving, cost reducing technical changes appear to have the unique and unambiguous effect of causing the equilibrium rate of profit to fall, and these effects are persistent across multiple sets of equations modelling output and price dynamics. Within Marx's own system and given his assumption of interest rates which float according to the capitalist savings, his claims are redeemed both in the case of disequalibrium continuous changes and in the case of a lower profit rate upon the re-assertion of equilibrium. Within the more correct-to-the-historical-moment assumption of fixed interest rates, we find that the falling rate of profit theory is alive and well, but only as a pure disequilibrium theory of continuous changes.

We thus find the theory of a technologically induced rate of profit to be empirically resurrected. However, this victory comes at the cost of admitting that the classical value system is insufficient to explaining these profit rate dynamics. In order to understand their effects, one must look away from the distribution of labor time and instead at the distribution of money capital investment. This can be seen as a consequence of the transformation problem, which witnesses that the logic of capitalism interprets values through the distorted lens of exploitation, allowing the price system to adjust in ways which cannot be predicted from the value system alone. Indeed, we have witnessed that the value rate of profit and the value composition of capital can diverge from their money equivalents. The classical system of labor value remains a valuable lens for understanding the real system of exploitation which prevails through the money mediator, and likely has practical value in the economics of post-capitalist societies, but it cannot be seen as a valid way to predict the dynamics of capitalism on its own.

The purpose of this paper has not been to prove formally when or why the rate of profit will fall in response to CULS technical change, but rather to prove that the empirical phenomenon is very real and that there is a great deal of work to be done in studying these effects within complex dynamic economic models. I find these results very convincing, and my hope is that through interacting with the software I have provided, readers can find themselves just as convinced. While I intend to continue the work of formally proving some of these results, there is clearly an entire theory here which deserves more development and investigation than I as a trained logician can be expected to handle on my own.

We are well on our way towards a fully revitalized 21st Marxist economic theory in which the old controversies of the falling rate of profit and the transformation problem have been resolved by working *through them* rather than avoiding and working around them.

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